

Essays on Supersolutions of BSDEs and Equilibrium Pricing in Generalized Capital Asset Pricing Models

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Für Ella und meine Eltern.

Abstract

In this thesis we study supersolutions of backward stochastic differential equations (BSDEs) and equilibrium pricing within two specific generalized capital asset pricing models (CAPMs).

In the first part of the thesis we begin by assuming that the generators of the BSDEs under consideration are jointly lower semicontinuous, bounded from below by an affine function of the control variable, and satisfy a specific normalization property. In our first main result we prove the existence and uniqueness of the minimal supersolution making use of a particular kind of semimartingale convergence and a suitably defined preorder on the set of supersolutions in combination with Zorn's lemma. In addition, we discuss possible relaxations of the assumptions imposed on the generator and extend our results from Brownian motion to arbitrary continuous local martingales.

Next, we assume the generators to be convex and introduce constraints to our setting by restricting the admissible controls to continuous semimartingales Z of the form $Z = z + \int \Delta du + \int \Gamma dW$ and allowing for a dependence of the generator on the respective decomposition parts. We introduce a notion of constrained minimality and then prove existence of supersolutions that are minimal at finitely many fixed times within the class where controls coincide up to these times. Besides providing stability results for the non-linear operator $\mathcal{E}_0^g(\cdot)$ that maps a terminal condition to the value of the minimal supersolution at time zero, we give a dual representation of $\mathcal{E}_0^g(\cdot)$, including an explicit computation of the conjugate in the case of a quadratic generator, and derive conditions for the existence of solutions under constraints by means of the duality results.

In the second part of the thesis we study equilibrium pricing in continuous time within affine and information-based CAPMs. Our model comprises finitely many economic agents and tradable securities. The agents seek to maximize exponential utilities and their endowments are spanned by the securities. In our first main result we show that an equilibrium exists and the agents' optimal trading strategies are constant and dependent on their respective risk aversion and endowment. In a next step, affine processes, and the theory of information-based asset pricing are used to model the endogenous asset price dynamics and the terminal payoff. Within both setups we derive semi-explicit pricing formulae which lend themselves to efficient numerical computations. In particular, no Monte Carlo methods are needed. Finally, we numerically analyze the impact of crucial parameters such as the agents' risk aversion or the intensity of jumps in the underlying's price on the implied volatility of simultaneously-traded European-style options within the affine framework, and investigate the dependence of credit-risky securities on the value of information about the financial standing of a company within the information-based framework.

Zusammenfassung

In dieser Arbeit untersuchen wir Superlösungen stochastischer Rückwärtsdifferentialgleichungen (BSDEs) und ein Gleichgewichtsmodell angewandt auf zwei spezifische verallgemeinerte “Capital Asset Pricing Models” (CAPMs).

Im ersten Teil der Arbeit fordern wir zunächst, dass die Generatoren der BSDEs gemeinsam unterhalbstetig und von unten durch eine affine Funktion der Kontrollvariablen beschränkt sind sowie eine spezifische Normalisierungseigenschaft erfüllen. In unserem ersten Hauptresultat beweisen wir Existenz und Eindeutigkeit der minimalen Superlösung, wobei wir eine spezielle Art der Semimartingalkonvergenz sowie eine geeignet definierte Präorder auf der Menge aller Superlösungen in Verbindung mit dem Zornschen Lemma nutzen. Zudem behandeln wir mögliche Abschwächungen unserer Anfangsannahmen an den Generator und weiten unsere Resultate von der Brownschen Bewegung auf allgemeine stetige lokale Martingale aus.

In einem nächsten Schritt nehmen wir an, dass die Generatoren konvex sind und führen Nebenbedingungen in unserem Rahmen ein, indem wir unsere admissiblen Kontrollen auf stetige Semimartingale Z restringieren, die eine Zerlegung der Form $Z = z + \int \Delta du + \int \Gamma dW$ erlauben, und eine Abhängigkeit des Generators von den jeweiligen Zerlegungsteilen zulassen. Wir definieren den Begriff der Minimalität unter Nebenbedingungen und beweisen dann die Existenz von Superlösungen, die an endlich vielen Zeitpunkten minimal sind innerhalb der Klasse, in der Kontrollprozesse bis zu diesen Zeitpunkten übereinstimmen. Neben Stabilitätsresultaten für den nichtlinearen Operator $\mathcal{E}_0^g(\cdot)$, der einer Endbedingung den Wert der minimalen Superlösung zum Zeitpunkt null zuordnet, leiten wir eine duale Darstellung von $\mathcal{E}_0^g(\cdot)$ her und geben eine explizite Form der Konjugierten im Falle eines quadratischen Generators an. Ferner präsentieren wir Bedingungen für die Existenz von Lösungen unter Nebenbedingungen mittels Nutzung der Dualitätsresultate.

Im zweiten Teil der Arbeit behandeln wir ein Gleichgewichtsmodell in stetiger Zeit für affine und sogenannte “information-based” CAPMs. Unser Modell umfasst endlich viele Agenten und handelbare Finanzprodukte. Die Agenten maximieren exponentielle Nutzenfunktionen und ihre Anfangsausstattung wird von den gehandelten Produkten aufgespannt. In unserem ersten Hauptresultat zeigen wir die Existenz eines Gleichgewichts, in welchem die optimalen Handelsstrategien der Agenten konstant sind und von ihrer jeweiligen Risikoaversion und Anfangsausstattung abhängen. Hiernach wird die Theorie der affinen Prozesse auf der einen Seite sowie die Theorie des sogenannten “Information-based Asset Pricing” auf der anderen Seite genutzt, um die endogene Preisdynamik der Assets und ihre Endauszahlung zu modellieren. Innerhalb beider Modelle leiten wir semi-explizite Preisformeln her, die sich für effiziente numerische Berechnungen eignen, da insbesondere keine Monte-Carlo-Methoden gebraucht werden. Schlussendlich analysieren wir zum einen im affinen Modell numerisch den Einfluss wichtiger Parameter, so wie zum Beispiel der Risikoaversion der Agenten oder der Sprungintensität im Preis des Underlyings, auf die implizite Volatilität simultan gehandelter europäischer Optionen. Zum anderen untersuchen wir numerisch die Abhängigkeit mit Ausfallrisiko behafteter Finanzprodukte vom Informationsgehalt über die finanzielle Lage eines Unternehmens.

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Introduction

In this thesis we study general as well as constrained supersolutions of a backward stochastic differential equation (henceforth BSDE) and a framework for equilibrium pricing in a generalized capital asset pricing model (henceforth CAPM), more precisely in an affine and an information-based CAPM.

Let us first elaborate on the subject of BSDEs and begin with a formal definition. A solution of a BSDE with generator g and terminal condition ξ is a pair of adapted processes (Y, Z) such that

$$Y_s - \int_s^t g_u(Y_u, Z_u) du + \int_s^t Z_u dW_u = Y_t, \quad Y_T = \xi, \quad (\text{I})$$

is satisfied for all times $0 \leq s \leq t \leq T$. Here, $T > 0$ is a fixed time horizon, W a d -dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and ξ is supposed to be an \mathcal{F}_T -measurable random variable. The process Y is in general referred to as the value process, whereas Z is usually called the control process. As a very basic motivating example corresponding to the particular generator $g \equiv 0$, consider the statement of the well-known martingale representation theorem in a Brownian filtration, compare for instance Revuz and Yor [60, Chapter V, Proposition 3.2]: for any square integrable, \mathcal{F}_T -measurable random variable ξ , there exists a unique square integrable process Z such that setting $Y_t := E[\xi | \mathcal{F}_t]$, for $t \in [0, T]$, it holds

$$Y_t = E[\xi] + \int_0^t Z_u dW_u.$$

Hence, since $Y_T = \xi$ by definition, (Y, Z) constitutes a solution of the BSDE with zero-generator and terminal condition ξ .

Let us now give a brief selection of the work done so far in the field. Whereas equations of the form (I) with linear generator were already introduced within a stochastic control framework in Bismut [7], the first rigorous study of the case of non-linear generators was given in the seminal work Pardoux and Peng [54]. There, the authors provide existence and uniqueness results for solutions of (I) given that the terminal condition is square integrable and the generator is Lipschitz continuous. This initial breakthrough spurred an immense interest in research of BSDEs which is still ongoing, resulting in an extensive literature on the subject. This is not surprising, as BSDEs constitute a powerful mathematical tool and find application in various fields of stochastic analy-

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sis and mathematical finance. Indeed, they are related among other to dynamic risk measures, to the problem of utility maximization and hedging in financial markets, to stochastic games and equilibria, in a Markovian framework to solutions of semilinear parabolic partial differential equations, or to more theoretical questions such as Skorokhod's embedding problem. Let us in particular point out the comprehensive work El Karoui et al. [30] as a milestone concerning the relevance of BSDEs within applications in mathematical finance, which in addition offered new results. Since solutions of BSDEs are often constructed as a fixed point of some suitable contraction mapping using a priori estimates, the Lipschitz assumption on the generator is crucial in many of the earlier works on the subject and it is certainly a reasonable setting for various application in finance, as indicated in El Karoui et al. [30]. Lipschitz continuity, however, is on the other hand too restrictive for many problems such as for instance that of utility maximization. The seminal paper Kobylanski [52], where the Lipschitz assumption is dropped and relaxed to that of quadratic growth in the control variable, was thus a landmark in the development of BSDE theory, enabling the further study of the important class of quadratic BSDEs. Initial restrictions such as bounded terminal conditions were gradually relaxed, as for example in Briand and Hu [8], and in general a considerable amount of research has been done in order to weaken impositions on the generator and terminal condition while still obtaining existence, uniqueness, stability or comparison results. That the case of a quadratic generator constitutes, however, a specific boundary to the well-posedness and solvability of BSDEs was shown in Delbaen et al. [22]. Indeed, the authors prove that solutions of BSDEs the generator of which exhibits superquadratic growth need not exist and if there exists a solution, then there automatically exist infinitely many. Hence, there is reason to extend the concept of solutions of BSDEs to that of supersolutions of BSDEs and since these are typically not unique, to study in particular minimal supersolutions. A supersolution of a BSDE with generator g and terminal condition ξ is a pair of adapted processes (Y, Z) such that the system of inequalities

$$Y_s - \int_s^t g_u(Y_u, Z_u) du + \int_s^t Z_u dW_u \geq Y_t, \quad Y_T \geq \xi, \quad (\text{II})$$

is satisfied for all times $0 \leq s \leq t \leq T$. Passing from equalities to inequalities, the value process Y is now naturally assumed to be càdlàg. Let us emphasize that also from an application point of view the step from solutions to supersolutions is essential, since for instance perfect hedging of a contingent claim ξ in an incomplete financial market is most often not possible, while superhedging is. Indeed, as with the one-to-one relation of the initial value of a solution of a BSDE to the unique arbitrage-free price of a contingent claim ξ in a complete market, so corresponds the initial value of the minimal supersolution to the superhedging price of ξ in an incomplete market.

The concept of supersolutions already briefly appears in El Karoui et al. [30] without an existence or uniqueness result and is then later treated more extensively in Peng [56], where the author studies existence and uniqueness of minimal supersolutions assuming

a Lipschitz generator and a square integrable terminal condition. In Drapeau et al. [24], the authors established existence, uniqueness and stability results for the minimal supersolution of a BSDE under rather weak assumptions on the generator and terminal condition. Indeed, they assumed ξ to be only integrable and g to be jointly lower semicontinuous, monotone in y , convex in z , and bounded below by an affine function of the control variable z . In particular, this setting allows for unbounded terminal conditions and discontinuous generators with non-Lipschitz growth in y and superquadratic growth in z . However, since the main proof strongly relies on compactness results given in Delbaen and Schachermayer [19], the convexity assumption on the generator is indispensable in the aforementioned work.

We take this as a starting point for the first chapter of this thesis, which aims at dropping the convexity, as well as the monotonicity. Thereby, our main mathematical contribution of Chapter 1 consists in establishing the existence of a unique minimal supersolution under the weakest possible assumptions by means of techniques and methods that are, to the best of our knowledge, new within the study of BSDEs. Specifically, all we need for our methods to work is a generator that is jointly lower semicontinuous, bounded from below by an affine function of the control variable, and satisfies a standard normalization property. As to a more detailed and precise description of our novel approach and the results obtained in Chapter 1, we refer the reader to Section 1.1.

While Chapter 1 is devoted to the derivation of existence and uniqueness of minimal supersolutions in a general, that is unconstrained setting, Chapter 2 in turn is dedicated to the question of how additional constraints on the admissible control processes alter our setup and in particular our notion of minimality. More precisely, we first restrict our admissible controls to continuous semimartingales of the form $Z = z + \int \Delta dt + \Gamma dW$, while then allowing the generator, which now is assumed to be jointly convex, to depend on the single decomposition parts. This accounts for the expression “delta and gamma constraints”. and the formulation of the problem (II) under constraints hence becomes

$$\begin{aligned} Y_s - \int_s^t g_u(Y_u, Z_u, \Delta_u, \Gamma_u) du + \int_s^t Z_u dW_u &\geq Y_t, \quad Y_T \geq \xi, \\ Z_t &= z + \int_0^t \Delta_u du + \int_0^t \Gamma_u dW_u. \end{aligned} \tag{III}$$

A supersolution in this context is a quadruplet of processes (Y, Z, Δ, Γ) satisfying (III) for all times $0 \leq s \leq t \leq T$. The consequence of this is twofold. First, we have given additional structure to the set of controls, making it furthermore possible to reflect a penalization of rapid changes of control values, as observed for instance in high frequency trading, by imposing a certain growth condition on g in the decomposition parts. Let us also stipulate that the above framework is flexible enough to encompass short selling constraints. Second, there is a need to adjust our notion of minimality. While the unconstrained problem as considered in Chapter 1 is remarkably stable with respect to concatenating supersolutions, an important property extensively used in the

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proofs, pasting two controls at arbitrary times within the current setup would violate our constraints. Consequently, comparing value processes at some time $t \in [0, T]$ has to be done subject to the corresponding control processes coinciding up to that time. Finding elements among all supersolutions that are minimal in the above sense amounts to a non-standard optimization problem with the leitmotif that minimality always is to be understood “given what one has done so far”. Having established the existence of supersolutions minimal at finitely many times, we derive in a next step stability properties of the non-linear operator $\mathcal{E}_0^g(\cdot)$ that maps a terminal condition to the value of the minimal supersolution at time zero, such as monotone convergence, Fatou’s lemma or L^1 -lower semicontinuity. This, together with the convexity, opens the door to studying the dual problem corresponding to the minimal initial value of supersolutions under the assumption that the generator is independent of y . It thereby provides additional insight into the structure of the problem, even allowing for an explicit computation of the conjugate in the case of a quadratic generator, and gives way to establishing conditions under which solutions of the BSDE under constraints exist.

Let us briefly discuss the literature on the subject. While there are numerous works on optimization or (super-)replication under constraints, see for instance Cvitanic and Karatzas [17], Jouini and Kallal [44] or Broadie et al. [9], so-called gamma constraints, where a bound to the diffusion part of the controls is imposed and which, among others, can easily be incorporated into our framework, were first introduced in Soner and Touzi [63] and then further investigated in Cheridito et al. [14]. Finally, we refer the reader to Section 2.1 for a more detailed and technical introduction to the setting.

The second part of this thesis is devoted to the field of equilibrium pricing in continuous time. Let us begin with a short motivation including a selective presentation of the literature on the subject. Ever since structured derivatives appeared on financial markets, initially as hedging instruments or measures to transfer a non-tradable risk to the markets, their variety and complexity increased considerably. Hence, providing models that enable us to price such instruments and that ideally are still tractable and therefore lend themselves to numerical calculations, is an essential task in today’s research in mathematical finance. Besides classical methods such as pricing by means of no-arbitrage or (super-)replication principles, one possible approach to price derivatives is the theory of equilibrium. This theory dates back to as early as the late 19th century and is based on the idea that in an economy comprising a certain number of economic agents, the price of an economic good can be determined according to the laws of supply and demand when all agents seek to maximize their respective utility of holding a specific amount of the good. Originally a static and deterministic theory it was steadily extended including concepts as contingent goods or incomplete financial markets. Here the seminal works Arrow [1] and Radner [59] were groundbreaking. Whereas the equilibrium theory in complete markets is well-understood and equilibria are characterized as Pareto-optimal allocations, the situation is more subtle when dropping the assumption of market completeness. Indeed, equilibria may fail to exist, compare for instance Hart [34], and need not be Pareto optimal. Thus the study of equilibria in incomplete mar-

kets is always restrained to special cases: For instance, a multiple agent model where markets are complete in equilibrium is considered in Karatzas et al. [47], whereas a specific class of goods and a particular choice of preferences is studied in Jofre et al. [43] and Carmona et al. [12], respectively. We refer the reader to Section 3.1 concerning a further selection of literature on the subject.

Recently, in Cheridito et al. [16] the authors derived existence and uniqueness of equilibrium results for incomplete financial market models in discrete time assuming the agents' preferences to be translation invariant. Furthermore, within a generalized CAPM, that is when all agents share the same base preferences (think for instance of exponential utilities) and endowments lie within the span of the tradable assets, aggregation methods can be used to reduce the multidimensional optimization problem to a single optimization of some representative agent, resulting in the agents sharing the market portfolio according to their respective risk aversions in equilibrium.

We take this as a starting point for Chapter 3 and first extend the result above to continuous time by means of duality methods. We thus consider a model with a finite number of agents, which are initially endowed with an attainable random payoff. The agents seek to maximize their expected exponential utility from terminal wealth by trading a finite number of financial securities that are characterized by their terminal payoffs and that span the agents' endowments. We assume the payoffs in turn to be functions of finitely many market factors that may or may not be observable to the agents. Specifically, we obtain that the equilibrium price process is fully characterized by the aggregated endowment, the market risk aversion, and the flow of market information. Our main contribution then consists in combining the obtained equilibrium pricing kernel with the framework of affine processes on the one hand, and with that of information-based asset pricing as established in Brody et al. [11] on the other hand, in order to derive (semi-)explicit equilibrium pricing formulae. Affine processes play a major role in mathematical finance, see for instance Duffie et al. [29], and are remarkable, because while still being tractable they cover a wide range of stochastic processes such as Lévy processes, Ornstein-Uhlenbeck processes, or Heston's stochastic volatility model, which may thus all be incorporated into our approach. Within the affine framework, we derive that equilibrium securities prices are given by the quotient of two integrals both of which are the product of an exponential function evaluated at the current state of the market factor process and the Fourier transform of a smooth function. In addition we study option implied volatilities within two single-security benchmark models, namely an additive Heston stochastic volatility model and a pure jump Ornstein-Uhlenbeck model. Information-based asset pricing in turn is perfectly tailored to the explicit construction of market information with the possibility of distinguishing between "genuine" information and market noise. Indeed, the market filtration is modeled by stochastic processes, which carry information about the a priori distribution of the market factors on the one hand, and embody pure noise preventing the agents from accessing full knowledge of the prevailing "true" value of the security at any time before the terminal cash flow occurs. Within the information-based framework, we prove that equilibrium dynamics are explicitly characterized by the conditional expectation of the terminal

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cash flows with respect to the equilibrium pricing kernel, given the partial information about the market factors that is available to the agents. Moreover, we show how the so-called innovations process appearing in classical filtering theory is connected to the equilibrium dynamics and, for a specific a priori distribution and payoff function, we illustrate how the equilibrium price can be worked out explicitly.

In particular, both our approaches allow for a fast and efficient numerical computation, since they are based on Fourier transform methods and thus no Monte-Carlo simulations are needed. First, we make use of that to analyze the impact of market risk aversion, stochastic volatility, or the intensity of jumps in the underlying's price on the implied volatility of simultaneously-traded European-style options within the affine framework. Second, we investigate how the “noisyness” of information about the financial standing of a company affects the prices of credit-risky securities within the information-based framework. For a more accurate introduction to Chapter 3 we refer the reader to Section 3.1.

This thesis is structured as follows. Chapter 1 is devoted to existence results of minimal supersolutions of BSDEs where we abstain from requiring any convexity assumptions on the generator. Setting and notations are specified in Section 1.2. A precise definition of minimal supersolutions and important structural properties, along with the main existence theorem, can then be found in Sections 1.3.1 and 1.3.2. Possible relaxations on the assumptions imposed on the generator are discussed in Section 1.3.3. Finally, Chapter 1 is concluded with a generalization of the results to the case of arbitrary continuous local martingales in Section 1.3.4.

In Chapter 2 in turn we assume the generators to be convex and add additional constraints on the set of admissible control processes. More precisely, we restrict our focus to continuous semimartingales Z allowing for a decomposition of the form $Z = z + \int \Delta du + \int \Gamma dW$, accounting for the notion of “delta and gamma constraints”. We then study existence and duality results of minimal supersolutions in this framework using a notion of minimality that is consistent with the constraints. A precise definition of supersolutions under gamma and delta constraints and minimality in this framework is given in Section 2.2, along with the main existence and stability results. Finally, Section 2.3 provides duality results along with explicit computations for a particular generator, and links the duality to the existence of solutions under constraints.

A theory of equilibrium pricing in two particular generalized CAPMs, more precisely in an affine and an information-based CAPM, is presented in Chapter 3. Assuming the agents have exponential utility functions and the individual endowments are spanned by the traded securities, we prove that an equilibrium exists and the agents' optimal trading strategies are constant. Moreover, it is shown that equilibrium price processes depend only on the aggregated endowment, the market risk aversion, and the flow of information. In Section 3.2 we give our general existence of equilibrium result along with a discussion on the information-generating processes. In Sections 3.3 and 3.4 we then present the affine and information-based pricing models, respectively. A brief addendum to regular affine processes can be found in the appendix.

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The content of this thesis is strongly based on Heyne et al. [38], Heyne et al. [39] and Horst et al. [41].

Part I.

Supersolutions of BSDEs

1. Minimal Supersolutions of BSDEs with Lower Semicontinuous Generators

1.1. Introduction

On a filtered probability space, the filtration of which is generated by a d -dimensional Brownian motion, we give conditions ensuring that the set $\mathcal{A}(\xi, g)$, consisting of all supersolutions (Y, Z) of a backward stochastic differential equation with terminal condition ξ and generator g , has a minimal element. Recall that a supersolution can be seen as, compare for instance El Karoui et al. [30], Peng [56], Drapeau et al. [24], a càdlàg value process Y and a control process Z , such that, for all $0 \leq s \leq t \leq T$,

$$Y_s - \int_s^t g_u(Y_u, Z_u) du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi$$

is satisfied.

Our ansatz to find the minimal supersolution is partially inspired by the methods and the setting introduced in Drapeau et al. [24]. More precisely, we start by considering the process $\hat{\mathcal{E}}^g(\xi)$, defined by

$$\hat{\mathcal{E}}_t^g(\xi) = \text{ess inf} \{ Y_t \in L^0(\mathcal{F}_t) : (Y, Z) \in \mathcal{A}(\xi, g) \}, \quad t \in [0, T].$$

It was shown in Drapeau et al. [24] that under a positivity assumption on the generator - this can be relaxed to a linear bound from below - the process $\hat{\mathcal{E}}^g(\xi)$ is in fact a supermartingale. Moreover, under such an assumption, given an adequate space of control processes, it follows that every value process of a supersolution is also a supermartingale. This is one of the key features of the approach in Drapeau et al. [24], and we will also adhere to the concept of supermartingale supersolutions. It allows us to consider the process $\mathcal{E}^g(\xi) = \lim_{s \downarrow \cdot, s \in \mathbb{Q}} \hat{\mathcal{E}}_s^g(\xi)$ as a candidate for the value process of the minimal supersolution.

Now, given this candidate value process, one needs to find a candidate control process \hat{Z} such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. In Drapeau et al. [24] this was done by constructing a monotone decreasing sequence of supersolutions converging to $\mathcal{E}^g(\xi)$ and by drawing on compactness results for sequences of martingales given in Delbaen and Schachermayer [19]. Owing to this approach, it was possible to characterize the candidate control pro-

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cess as the limit of a sequence of convex combinations of control processes. Therefore, in order to verify that the pair $(\mathcal{E}^g(\xi), \hat{Z})$ is a supersolution, it was crucial that the generator is convex with respect to the control variable. The principal aim of this chapter is to drop this convexity assumption.

In order to obtain the existence of a minimal supersolution without taking convex combinations, we proceed as follows. Our first idea is to use results on semimartingale convergence given in Barlow and Protter [4]. Loosely speaking, given a sequence of special semimartingales that converges uniformly, in some sense to be made precise, to some limit process, their result guarantees that the limit process is also a special semimartingale and that the locale martingale parts converge in \mathcal{H}^1 to the local martingale in the decomposition of the limit process. Interpreted in our setting, this implies that, if we can construct a sequence $((Y^n, Z^n))$ of supersolutions such that (Y^n) converges in the \mathcal{R}^∞ -norm to $\mathcal{E}^g(\xi)$, then we obtain the existence of a candidate control process \hat{Z} as the limit of the sequence (Z^n) .

Now, our second main idea shows how to construct a sequence converging in the sense of Barlow and Protter [4]. To that end, we prove that, for $\varepsilon > 0$, there exists $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{A}(\xi, g)$ such that $\|Y^\varepsilon - \mathcal{E}^g(\xi)\|_{\mathcal{R}^\infty} \leq \varepsilon$. Note that it is not possible to infer the existence of such a supersolution from the approach taken in Drapeau et al. [24], where the approximating sequence was decreasing, but only uniform on a finite set of rationals. Therefore, we have to develop a new method. The central idea is to define a suitable preorder on the set of supersolutions and to use Zorn's lemma to show the existence of a maximal element. To set up our preorder, we associate with each supersolution (Y, Z) the stopping time τ , at which Y first leaves the ε -neighborhood of $\mathcal{E}^g(\xi)$. With this at hand, we say (Y^1, Z^1) dominates (Y^2, Z^2) , if and only if $\tau^1 \geq \tau^2$ and the processes coincide up to τ^2 . Given this preorder, we have to show that each totally ordered chain has an upper bound. In order to achieve this, we assume a mild normalization condition on the generator. In its simplest form it states that g equals zero as soon as the control variable is zero. This assumption is well known especially in the context of g -expectations, see for example Peng [55] and Drapeau et al. [24]. More generally, we ask for a certain very simple SDE to have a solution on some short time interval. Combining this assumption with the supermartingale structure of our setting, in particular with arguments based on supermartingale convergence, yields the existence of an upper bound. Moreover, we can show that the stopping time associated with the maximal element provided by Zorn's lemma equals T .

The previous arguments show that we obtain indeed a pair of candidate processes $(\mathcal{E}^g(\xi), \hat{Z})$. It remains to verify that the candidate pair is an element of $\mathcal{A}(\xi, g)$. However, this is straightforward by assuming that the generator is jointly lower semicontinuous and can be done by similar arguments as in Drapeau et al. [24].

Let us briefly discuss the existing literature on related problems, a broader discussion of which can be found in Drapeau et al. [24]. Nonlinear BSDEs were first introduced in Pardoux and Peng [54]. In this seminal work existence and uniqueness results were given for the case of Lipschitz generators and square integrable terminal conditions. Kobylanski [52] studies BSDEs with quadratic generators, whereas Delbaen et al. [22]

consider superquadratic BSDEs with positive generators that are convex in z and independent of y . BSDEs with generators that are not locally Lipschitz are studied in Bahlali et al. [2]. Among the first introducing supersolutions of BSDEs were El Karoui et al. [30, Section 2.3]. Further references can also be found in Peng [56], who studies the existence and uniqueness of constrained minimal supersolutions under the assumption of a Lipschitz generator and square integrable terminal conditions. For a link between minimal supersolutions of BSDEs and solutions of reflected BSDEs see Peng and Xu [57]. Most recently, Cheridito and Stadje [13] have analyzed existence and stability of supersolutions of BSDEs. They consider terminal conditions which are functionals of the underlying Brownian motion and generators that are convex in z and Lipschitz in y , and they work with discrete time approximations of BSDEs. Furthermore, the concept of supersolutions is closely related to Peng's g -expectations, see for instance Drapeau et al. [24], Peng [55], since the mapping $\xi \mapsto \mathcal{E}_0^g(\xi)$ can be seen as a nonlinear expectation.

The chapter is organized as follows. Setting and notations are specified in Section 1.2. A precise definition of minimal supersolutions and important structural properties of $\hat{\mathcal{E}}^g(\xi)$, along with the main existence theorem, can then be found in Sections 1.3.1 and 1.3.2. Possible relaxations on the assumptions imposed on the generator are discussed in Section 1.3.3. Finally, we conclude the chapter with a generalization of our results to the case of arbitrary continuous local martingales in Section 1.3.4.

1.2. Setting and Notations

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, where the filtration (\mathcal{F}_t) is generated by a d -dimensional Brownian motion W and is assumed to satisfy the usual conditions. For some fixed time horizon $T > 0$ and for all $t \in [0, T]$, the sets of \mathcal{F}_t -measurable random variables are denoted by $L^0(\mathcal{F}_t)$, where random variables are identified in the P -almost sure sense. Let furthermore denote $L^p(\mathcal{F}_t)$ the set of random variables in $L^0(\mathcal{F}_t)$ with finite p -norm, for $p \in [1, +\infty]$. Inequalities and strict inequalities between any two random variables or processes X^1 and X^2 are understood in the P -almost sure or in the $P \otimes dt$ -almost everywhere sense, respectively. We denote by \mathcal{T} the set of stopping times with values in $[0, T]$ and hereby call an increasing sequence of stopping times (τ^n) such that $P[\bigcup_n \{\tau^n = T\}] = 1$ a localizing sequence of stopping times. By $\mathcal{S} := \mathcal{S}(\mathbb{R})$ we denote the set of càdlàg progressively measurable processes Y with values in \mathbb{R} . For $p \in [1, +\infty[$, we further denote by \mathcal{H}^p the set of càdlàg local martingales M with finite \mathcal{H}^p -norm on $[0, T]$, that is $\|M\|_{\mathcal{H}^p} := E[\langle M, M \rangle_T^{p/2}]^{1/p} < \infty$. By $\mathcal{L}^p := \mathcal{L}^p(W)$ we denote the set of $\mathbb{R}^{1 \times d}$ -valued, progressively measurable processes Z such that $\int Z dW \in \mathcal{H}^p$, that is, $\|Z\|_{\mathcal{L}^p} := E[(\int_0^T |Z_s|^2 ds)^{p/2}]^{1/p}$ is finite, where $|\cdot|$ denotes the Euclidian norm. For $Z \in \mathcal{L}^p$, the stochastic integral $\int Z dW$ is well defined, see Protter [58], and is by means of the Burkholder-Davis-Gundy inequality (Protter [58, Theorem 48]) a continuous martingale. We further denote by $\mathcal{L} := \mathcal{L}(W)$ the set of $\mathbb{R}^{1 \times d}$ -valued, progressively measurable processes Z such that there exists a localizing sequence of stopping times (τ^n) with $Z1_{[0, \tau^n]} \in \mathcal{L}^1$, for all $n \in \mathbb{N}$. For $Z \in \mathcal{L}$, the

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stochastic integral $\int Z dW$ is well defined and is a continuous local martingale. Furthermore, for a process X , let X^* denote the following expression $X^* := \sup_{t \in [0, T]} |X_t|$, by which we define the norm $\|X\|_{\mathcal{R}^\infty} := \|X^*\|_{L^\infty}$.

We call a càdlàg semimartingale X a special semimartingale if it can be decomposed into $X = X_0 + M + A$, where M is a local martingale and A a predictable process of finite variation such that $M_0 = A_0 = 0$. Such a decomposition is then unique, compare for instance Protter [58, Chapter III, Theorem 30], and is called the canonical decomposition of X .

1.3. Minimal Supersolutions of BSDEs

1.3.1. First Definitions and Structural Properties

Throughout this chapter, a generator is a jointly measurable function g from $\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d}$ to $\mathbb{R} \cup \{+\infty\}$ where $\Omega \times [0, T]$ is endowed with the progressive σ -field. Given a generator g and a terminal condition $\xi \in L^0(\mathcal{F}_T)$, a pair $(Y, Z) \in \mathcal{S} \times \mathcal{L}$ is a supersolution of a BSDE, if, for $0 \leq s \leq t \leq T$, holds

$$Y_s - \int_s^t g_u(Y_u, Z_u) du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi. \quad (1.1)$$

For a supersolution (Y, Z) , we call Y the value process and Z its corresponding control process. Note that the formulation in (1.1) is equivalent to the existence of a càdlàg increasing process K , with $K_0 = 0$, such that

$$Y_t = \xi + \int_t^T g_u(Y_u, Z_u) du + (K_T - K_t) - \int_t^T Z_u dW_u, \quad t \in [0, T]. \quad (1.2)$$

Although the notation in (1.2) is standard in the literature concerning supersolutions of BSDEs, see for example El Karoui et al. [30] and Peng [56], we will work with (1.1), since the proof of our main result exploits this structure. A control process Z is said to be admissible if the continuous local martingale $\int Z dW$ is a supermartingale. Throughout this chapter a generator g is said to be

(LSC) if $(y, z) \mapsto g(\omega, t, y, z)$ is lower semicontinuous, for all $(\omega, t) \in \Omega \times [0, T]$.

(POS) positive if $g(y, z) \geq 0$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$.

(NOR) normalized if $g_t(y, 0) = 0$, for all $(t, y) \in [0, T] \times \mathbb{R}$.

We are now interested in the set

$$\mathcal{A}(\xi, g) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : Z \text{ is admissible and (1.1) holds}\} \quad (1.3)$$

and the process

$$\hat{\mathcal{E}}_t^g(\xi) := \text{ess inf} \{ Y_t \in L^0(\mathcal{F}_t) : (Y, Z) \in \mathcal{A}(\xi, g) \}, \quad t \in [0, T]. \quad (1.4)$$

A pair (Y, Z) is called minimal supersolution if $(Y, Z) \in \mathcal{A}(\xi, g)$, and if for any other supersolution $(Y', Z') \in \mathcal{A}(\xi, g)$, holds $Y_t \leq Y'_t$, for all $t \in [0, T]$.

For the proof of our main existence theorem we will need some auxiliary results concerning structural properties of $\hat{\mathcal{E}}^g(\xi)$ and supersolutions (Y, Z) in $\mathcal{A}(\xi, g)$.

Lemma 1.1. *Let g be a generator satisfying (POS). Assume further that $\mathcal{A}(\xi, g) \neq \emptyset$ and that for the terminal condition ξ holds $\xi^- \in L^1(\mathcal{F}_T)$. Then $\xi \in L^1(\mathcal{F}_T)$ and, for any $(Y, Z) \in \mathcal{A}(\xi, g)$, the control Z is unique and the value process Y is a supermartingale such that $Y_t \geq E[\xi | \mathcal{F}_t]$. Moreover, the unique canonical decomposition of Y is given by*

$$Y = Y_0 + M - A, \quad (1.5)$$

where $M = \int Z dW$ and A is an increasing, predictable, càdlàg process with $A_0 = 0$.

The proof of Lemma 1.1 can be found in Drapeau et al. [24, Lemma 3.2].

Proposition 1.2. *Suppose that $\mathcal{A}(\xi, g) \neq \emptyset$ and let $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. If g satisfies (POS), then the process $\hat{\mathcal{E}}^g(\xi)$ is a supermartingale. In particular,*

$$\mathcal{E}_t^g(\xi) := \lim_{s \downarrow t, s \in \mathbb{Q}} \hat{\mathcal{E}}_s^g(\xi), \quad \text{for all } t \in [0, T) \quad \text{and} \quad \mathcal{E}_T^g(\xi) := \xi$$

is a càdlàg supermartingale such that

$$\hat{\mathcal{E}}_t^g(\xi) \geq \mathcal{E}_t^g(\xi), \quad \text{for all } t \in [0, T].$$

Furthermore, the following two pasting properties hold true.

1. Let $(Z^n) \subset \mathcal{L}$ be admissible, $\sigma \in \mathcal{T}$, and $(B_n) \subset \mathcal{F}_\sigma$ be a partition of Ω . Then the pasted process $\bar{Z} = Z^1 1_{[0, \sigma]} + \sum_{n \geq 1} Z^n 1_{B_n} 1_{[\sigma, T]}$ is admissible.
2. Let $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g)$, $\sigma \in \mathcal{T}$ and $(B_n) \subset \mathcal{F}_\sigma$ be as before. If $Y_{\sigma-}^1 1_{B_n} \geq Y_\sigma^n 1_{B_n}$ holds true for all $n \in \mathbb{N}$, then $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$, where

$$\bar{Y} = Y^1 1_{[0, \sigma]} + \sum_{n \geq 1} Y^n 1_{B_n} 1_{[\sigma, T]} \quad \text{and} \quad \bar{Z} = Z^1 1_{[0, \sigma]} + \sum_{n \geq 1} Z^n 1_{B_n} 1_{[\sigma, T]}.$$

Proof. The proof of the part concerning the process $\mathcal{E}^g(\xi)$ can be found in Drapeau et al. [24, Proposition 3.4]. \bar{Z} is admissible by Drapeau et al. [24, Lemma 3.1.1]. We can approximate σ from below by some foretelling sequence of stopping times $(\eta_m)^1$,

¹Such a sequence satisfying $\eta_m \leq \eta_{m+1} < \sigma$, for all $m \in \mathbb{N}$, and $\lim_m \eta_m = \sigma$, always exists, since in a Brownian filtration every stopping time is predictable, compare Revuz and Yor [60, Corollary V.3.3].

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and then show, analogously to Drapeau et al. [24, Lemma 3.1.2], that the pair (\bar{Y}, \bar{Z}) satisfies Inequality (1.1) and is thus an element of $\mathcal{A}(\xi, g)$. \square

Remark 1.3. Whenever a stopping time σ takes values in a countable subset \mathcal{S} of $[0, T]$, the adapted process $\hat{\mathcal{E}}^g(\xi)$ evaluated at σ is defined by

$$\hat{\mathcal{E}}_\sigma^g(\xi) := \sum_{s \in \mathcal{S}} 1_{A_s} \hat{\mathcal{E}}_s^g(\xi) \quad \text{with } A_s := \{\sigma = s\}.$$

It is straightforward to show that $\hat{\mathcal{E}}_\sigma^g(\xi)$ is \mathcal{F}_σ -measurable and consistent with (1.4), in the sense that

$$\hat{\mathcal{E}}_\sigma^g(\xi) = \text{ess inf } \{Y_\sigma : (Y, Z) \in \mathcal{A}(\xi, g)\}.$$

Proposition 1.2 yields that the set $\{Y_\sigma : (Y, Z) \in \mathcal{A}(\xi, g)\}$ is directed downwards, see Drapeau et al. [24, Proposition 3.3.1], and as a consequence we can find, for any $\varepsilon > 0$, some $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{A}(\xi, g)$ such that

$$Y_\sigma^\varepsilon \leq \hat{\mathcal{E}}_\sigma^g(\xi) + \varepsilon. \quad \blacklozenge$$

Proposition 1.4. *Let $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots$ be a sequence of stopping times converging to the finite stopping time $\tau^* = \lim_{n \rightarrow \infty} \tau_n$. Further, let (Y^n) be a sequence of càdlàg supermartingales such that $Y_{\tau_n-}^n \geq Y_{\tau_n}^{n+1}$, and which satisfies $Y^n 1_{[\tau_{n-1}, \tau_n[} \geq M 1_{[\tau_{n-1}, \tau_n[}$, where M is a uniformly integrable martingale. Then, for any sequence of stopping times $\sigma_n \in [\tau_{n-1}, \tau_n[$, the limit $Y^\infty := \lim_{n \rightarrow \infty} Y_{\sigma_n}^n$ exists and the process*

$$\bar{Y} := \sum_{n \geq 1} Y^n 1_{[\tau_{n-1}, \tau_n[} + Y^\infty 1_{[\tau^*, \infty[}$$

is a càdlàg supermartingale. Moreover, the limit Y^∞ is independent of the approximating sequence $(Y_{\sigma_n}^n)$ and, if all Y^n are continuous and $Y_{\tau_n}^n = Y_{\tau_n}^{n+1}$, for all $n \in \mathbb{N}$, then \bar{Y} is continuous.

Proof. Note that $(Y_{\sigma_n}^n)$ is a (\mathcal{F}_{σ_n}) -supermartingale. Indeed, if $(\tilde{\eta}_m) \uparrow \tau_n$ is a foretelling sequence of stopping times, then, with $\eta_m := \tilde{\eta}_m \vee \tau_{n-1}$, the family $((Y_{\eta_m}^n)^-)_{m \in \mathbb{N}}$ is uniformly integrable and we obtain

$$\begin{aligned} E[Y_{\sigma_{n+1}}^{n+1} \mid \mathcal{F}_{\sigma_n}] &= E[E[Y_{\sigma_{n+1}}^{n+1} \mid \mathcal{F}_{\tau_n}] \mid \mathcal{F}_{\sigma_n}] \leq E[Y_{\tau_n}^{n+1} \mid \mathcal{F}_{\sigma_n}] \leq E[Y_{\tau_n-}^n \mid \mathcal{F}_{\sigma_n}] \\ &\leq \liminf_m E[Y_{\eta_m}^n \mid \mathcal{F}_{\sigma_n}] \leq \liminf_m Y_{\eta_m \wedge \sigma_n}^n = Y_{\sigma_n}^n. \end{aligned}$$

Moreover, $((Y_{\sigma_n}^n)^-)$ is uniformly integrable. Hence, the sequence $(Y_{\sigma_n}^n)$ converges by the supermartingale convergence theorem, see Dellacherie and Meyer [23, Theorems V.28, 29], to some random variable Y^∞ , P -almost surely, and thus \bar{Y} is well-defined. Furthermore, the limit Y^∞ is independent of the approximating sequence $(Y_{\sigma_n}^n)$. Indeed, for any other sequence $(\tilde{\sigma}_n)$ with $\tilde{\sigma}_n \in [\tau_{n-1}, \tau_n[$, the limit $\lim_n Y_{\tilde{\sigma}_n}^n$ exists by the same argumentation. Now $\lim_n Y_{\sigma_n}^n = \lim_n Y_{\tilde{\sigma}_n}^n = Y^\infty$ holds, since the sequence $(\hat{\sigma}_n)$ defined

by

$$\hat{\sigma}_n := \begin{cases} \sigma_{\frac{n}{2}} \vee \tilde{\sigma}_{\frac{n}{2}} & , \text{ for } n \text{ even} \\ \sigma_{\frac{n+1}{2}} \wedge \tilde{\sigma}_{\frac{n+1}{2}} & , \text{ for } n \text{ odd} \end{cases}$$

satisfies $\hat{\sigma}_n \in [\tau_{n-1}, \tau_n[$ and $\lim_n Y_{\hat{\sigma}_n}^n$ exists. Thus, all limits must coincide. Next, we show that \bar{Y}^{σ_n} is a supermartingale, for all $n \in \mathbb{N}$. To this end first observe that, for all $0 \leq s \leq t$,

$$\begin{aligned} E[\bar{Y}_t^{\sigma_n} - \bar{Y}_s^{\sigma_n} \mid \mathcal{F}_s] &= \sum_{k=0}^{n-2} E[E[\bar{Y}_{(\tau_{k+1} \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_k \vee s) \wedge t}^{\sigma_n} \mid \mathcal{F}_{(\tau_k \vee s) \wedge t}] \mid \mathcal{F}_s] \\ &\quad + E[E[\bar{Y}_{(\sigma_n \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_{n-1} \vee s) \wedge t}^{\sigma_n} \mid \mathcal{F}_{(\tau_{n-1} \vee s) \wedge t}] \mid \mathcal{F}_s] \\ &\quad + E[E[\bar{Y}_t^{\sigma_n} - \bar{Y}_{(\sigma_n \vee s) \wedge t}^{\sigma_n} \mid \mathcal{F}_{(\sigma_n \vee s) \wedge t}] \mid \mathcal{F}_s]. \end{aligned}$$

Note further that, for each $n \in \mathbb{N}$, the process \bar{Y}^{σ_n} is càdlàg and can only jump downwards, that is, $\bar{Y}_{t-}^{\sigma_n} \geq \bar{Y}_t^{\sigma_n}$, for all $t \in \mathbb{R}$. Observe to this end that, on the one hand, $\bar{Y}_{\tau_k-}^{\sigma_n} = Y_{\tau_k-}^k \geq Y_{\tau_k}^{k+1} = \bar{Y}_{\tau_k}^{\sigma_n}$, for all $0 \leq k \leq n-1$, by assumption, where we assumed $\tau_{k-1} < \tau_k$, without loss of generality. On the other hand, Y^k can only jump downwards. Indeed, as càdlàg supermartingales, all Y^k can be decomposed into $Y^k = Y_0^k + M^k - A^k$, by the Doob-Meyer decomposition theorem Protter [58, Chapter III, Theorem 13], where M^k is a local martingale and A^k a predictable, increasing process with $A_0^k = 0$. Since in a Brownian filtration every local martingale is continuous, the claim follows.

Thus, for all $0 \leq k \leq n-2$, and $(\tilde{\eta}_m) \uparrow \tau_{k+1}$ a foretelling sequence of stopping times, it holds with $\eta_m := \tilde{\eta}_m \vee \tau_k$,

$$\begin{aligned} E[\bar{Y}_{(\tau_{k+1} \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_k \vee s) \wedge t}^{\sigma_n} \mid \mathcal{F}_{(\tau_k \vee s) \wedge t}] &\leq E[\bar{Y}_{((\tau_{k+1}-) \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_k \vee s) \wedge t}^{\sigma_n} \mid \mathcal{F}_{(\tau_k \vee s) \wedge t}] \\ &= E[\liminf_m \bar{Y}_{(\eta_m \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_k \vee s) \wedge t}^{\sigma_n} \mid \mathcal{F}_{(\tau_k \vee s) \wedge t}] \\ &\leq E[\liminf_m Y_{(\eta_m \vee s) \wedge t}^{k+1} \mid \mathcal{F}_{(\tau_k \vee s) \wedge t}] - Y_{(\tau_k \vee s) \wedge t}^{k+1} \\ &\leq \liminf_m E[Y_{(\eta_m \vee s) \wedge t}^{k+1} \mid \mathcal{F}_{(\tau_k \vee s) \wedge t}] - Y_{(\tau_k \vee s) \wedge t}^{k+1} \leq 0. \end{aligned}$$

Moreover, $E[\bar{Y}_t^{\sigma_n} - \bar{Y}_{(\sigma_n \vee s) \wedge t}^{\sigma_n} \mid \mathcal{F}_{(\sigma_n \vee s) \wedge t}] = 0$, as well as

$$\begin{aligned} E[\bar{Y}_{(\sigma_n \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_{n-1} \vee s) \wedge t}^{\sigma_n} \mid \mathcal{F}_{(\tau_{n-1} \vee s) \wedge t}] &\leq E[Y_{(\sigma_n \vee s) \wedge t}^n - Y_{(\tau_{n-1} \vee s) \wedge t}^n \mid \mathcal{F}_{(\tau_{n-1} \vee s) \wedge t}] \leq 0. \end{aligned}$$

Combining this we obtain that $E[\bar{Y}_t^{\sigma_n} \mid \mathcal{F}_s] \leq \bar{Y}_s^{\sigma_n}$. Furthermore, $\lim_n \bar{Y}_t^{\sigma_n} = \bar{Y}_t$, for all $t \in \mathbb{R}$. Indeed, let us write $\lim_n \bar{Y}_t^{\sigma_n} = \lim_n \bar{Y}_t^{\sigma_n} 1_{\{t < \tau^*\}} + \lim_n \bar{Y}_t^{\sigma_n} 1_{\{t \geq \tau^*\}}$. Then, $\lim_n \bar{Y}_t^{\sigma_n} 1_{\{t \geq \tau^*\}} = \lim_n Y_{\sigma_n}^n 1_{\{t \geq \tau^*\}} = Y^\infty 1_{\{t \geq \tau^*\}} = \bar{Y}_t 1_{\{t \geq \tau^*\}}$ and $\lim_n \bar{Y}_{\sigma_n \wedge t}^{\sigma_n} 1_{\{t < \tau^*\}} = \bar{Y}_t 1_{\{t < \tau^*\}}$. Hence, the claim follows. As a consequence of Fatou's lemma it now holds

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that

$$E[\bar{Y}_t | \mathcal{F}_s] \leq \liminf_{n \rightarrow \infty} E[\bar{Y}_t^{\sigma_n} | \mathcal{F}_s] \leq \liminf_{n \rightarrow \infty} \bar{Y}_s^{\sigma_n} = \bar{Y}_s,$$

since the family $((\bar{Y}_t^{\sigma_n})^-)$ is uniformly integrable. Hence, \bar{Y} is a supermartingale, which by construction has right-continuous paths and Karatzas and Shreve [45, Theorem 1.3.8] then yields that \bar{Y} is even càdlàg. Finally, whenever all Y^n are continuous and $Y_{\tau_n}^n = Y_{\tau_n}^{n+1}$ holds, for all $n \in \mathbb{N}$, the process \bar{Y} is continuous per construction. \square

1.3.2. Existence and Uniqueness of Minimal Supersolutions

We are now ready to state our main existence result. Possible relaxations of the assumptions (POS) and (NOR) imposed on the generator are discussed in Section 1.3.3. Note that it is not our focus to investigate conditions assuring the crucial assumption that $\mathcal{A}(\xi, g) \neq \emptyset$. See Drapeau et al. [24] and the references therein for further details.

Theorem 1.5. *Let g be a generator satisfying (LSC), (POS) and (NOR) and $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. If $\mathcal{A}(\xi, g) \neq \emptyset$, then $\mathcal{E}^g(\xi)$ is the value process of the unique minimal supersolution, that is, there exists a unique control process $\hat{Z} \in \mathcal{L}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$.*

Observe that Theorem 1.5 and Proposition 1.2 imply that $\mathcal{E}^g(\xi)$ is a modification of $\hat{\mathcal{E}}^g(\xi)$.

Proof. Step 1: Uniform Limit and verification. Since $\mathcal{A}(\xi, g) \neq \emptyset$, there exist $(Y^b, Z^b) \in \mathcal{A}(\xi, g)$. From now on we restrict our focus to supersolutions (\bar{Y}, \bar{Z}) in $\mathcal{A}(\xi, g)$ satisfying $\bar{Y}_0 \leq Y_0^b$. Indeed, since we are only interested in minimal supersolutions, we can paste any value process of $(Y, Z) \in \mathcal{A}(\xi, g)$ at $\tau := \inf\{t > 0 : Y_t^b > Y_t\} \wedge T$ such that $\bar{Y} := Y^b 1_{[0, \tau[} + Y 1_{[\tau, T]}$ satisfies $\bar{Y}_0 \leq Y_0^b$, where the corresponding control \bar{Z} is obtained as in Proposition 1.2.

Assume for the beginning that we can find a sequence $((Y^n, Z^n))$ within $\mathcal{A}(\xi, g)$ such that

$$\lim_{n \rightarrow \infty} \|Y^n - \mathcal{E}^g(\xi)\|_{\mathcal{R}^\infty} = 0. \quad (1.6)$$

Since all Y^n are càdlàg supermartingales, they are, by the Doob-Meyer decomposition theorem, special semimartingales with canonical decomposition $Y^n = Y_0^n + M^n - A^n$ as in (1.5). The supermartingale property of all $\int Z^n dW$ and $\xi \in L^1(\mathcal{F}_T)$, compare Lemma 1.1, imply that $E[A_T^n] \leq Y_0^b - E[\xi] < \infty$. Hence, since each A^n is increasing, $\sup_n E[\int_0^T |dA_s^n|] < \infty$. As (1.6) implies in particular that $\lim_{n \rightarrow \infty} E[(Y^n - \mathcal{E}^g(\xi))^*] = 0$, it follows from Barlow and Protter [4, Theorem 1 and Corollary 2] that $\mathcal{E}^g(\xi)$ is a special semimartingale with canonical decomposition $\mathcal{E}^g(\xi) = \mathcal{E}_0^g(\xi) + M - A$ and that

$$\lim_{n \rightarrow \infty} \|M^n - M\|_{\mathcal{H}^1} = 0 \quad , \quad \lim_{n \rightarrow \infty} E[(A^n - A)^*] = 0.$$

The local martingale M is continuous and allows a representation of the form $M =$

$\int \hat{Z} dW$, where $\hat{Z} \in \mathcal{L}$, compare Protter [58, Chapter IV, Theorem 43]. Since

$$E \left[\left(\int_0^T (Z_u^n - \hat{Z}_u)^2 du \right)^{1/2} \right] \xrightarrow{n \rightarrow +\infty} 0,$$

we have that, up to a subsequence, (Z^n) converges $P \otimes dt$ -almost everywhere to \hat{Z} and $\lim_{n \rightarrow \infty} \int_0^t Z^n dW = \int_0^t \hat{Z} dW$, for all $t \in [0, T]$, P -almost surely, due to the Burkholder-Davis-Gundy inequality. In particular, $\lim_{n \rightarrow \infty} Z^n(\omega) = \hat{Z}(\omega)$, dt -almost everywhere, for almost all $\omega \in \Omega$.

In order to verify that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$, we will use the convergence obtained above. More precisely, for all $0 \leq s \leq t \leq T$, Fatou's lemma together with (1.6) and the lower semicontinuity of the generator yields

$$\begin{aligned} \mathcal{E}_s^g(\xi) - \int_s^t g_u(\mathcal{E}_u^g(\xi), \hat{Z}_u) du + \int_s^t \hat{Z}_u dW_u \\ \geq \limsup_n \left(Y_s^n - \int_s^t g_u(Y_u^n, Z_u^n) du + \int_s^t Z_u^n dW_u \right) \geq \limsup_n Y_t^n = \mathcal{E}_t^g(\xi). \end{aligned}$$

The above, the positivity of g and $\mathcal{E}^g(\xi) \geq E[\xi | \mathcal{F}]$ imply that $\int \hat{Z} dW \geq E[\xi | \mathcal{F}] - \mathcal{E}_0^g(\xi)$. Hence, being bounded from below by a martingale, the continuous local martingale $\int \hat{Z} dW$ is a supermartingale. Thus, \hat{Z} is admissible and $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$ and therefore, by Lemma 1.1, \hat{Z} is unique. Since we know by Proposition 1.2 that $\hat{\mathcal{E}}_t^g(\xi) \geq \mathcal{E}_t^g(\xi)$, for all $t \in [0, T]$, we deduce that $\hat{\mathcal{E}}_t^g(\xi) = \mathcal{E}_t^g(\xi)$, for all $t \in [0, T]$, by the definition of $\hat{\mathcal{E}}^g(\xi)$. Hence, $(\mathcal{E}^g(\xi), \hat{Z})$ is the unique minimal supersolution.

Step 2: A preorder on $\mathcal{A}(\xi, g)$. As to the existence of $((Y^n, Z^n))$ satisfying (1.6), it is sufficient to show that, for arbitrary $\varepsilon > 0$, we can find a supersolution $(Y^\varepsilon, Z^\varepsilon)$ satisfying

$$\|Y^\varepsilon - \mathcal{E}^g(\xi)\|_{\mathcal{R}^\infty} \leq \varepsilon. \quad (1.7)$$

We define the following preorder² on $\mathcal{A}(\xi, g)$

$$(Y^1, Z^1) \preceq (Y^2, Z^2) \Leftrightarrow \tau_1 \leq \tau_2 \text{ and } (Y^1, Z^1)1_{[0, \tau_1[} = (Y^2, Z^2)1_{[0, \tau_1[}, \quad (1.8)$$

where, for $i = 1, 2$,

$$\tau_i = \inf \{ t \geq 0 : Y_t^i > \mathcal{E}_t^g(\xi) + \varepsilon \} \wedge T. \quad (1.9)$$

²Note that, in order to apply Zorn's lemma, we need a partial order instead of just a preorder. To this end we consider equivalence classes of processes. Two supersolutions $(Y^1, Z^1), (Y^2, Z^2) \in \mathcal{A}(\xi, g)$ are said to be equivalent if $(Y^1, Z^1) \preceq (Y^2, Z^2)$ and $(Y^2, Z^2) \preceq (Y^1, Z^1)$. This means that they are equal up to their corresponding stopping time $\tau_1 = \tau_2$ as in (1.9). This induces a partial order on the set of equivalence classes and hence the use of Zorn's lemma is justified.

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For any totally ordered chain $((Y^i, Z^i))_{i \in I}$ within $\mathcal{A}(\xi, g)$ with corresponding stopping times τ_i , we want to construct an upper bound. If we consider

$$\tau^* = \operatorname{ess\,sup}_{i \in I} \tau_i,$$

we know by the monotonicity of the stopping times that we can find a monotone subsequence (τ_m) of $(\tau_i)_{i \in I}$ such that $\tau^* = \lim_{m \rightarrow \infty} \tau_m$. In particular, τ^* is a stopping time. Furthermore, the structure of the preorder (1.8) yields that the value processes of the supersolutions $((Y^m, Z^m))$ corresponding to the stopping times (τ_m) satisfy

$$Y_{\tau_m}^{m+1} \leq Y_{\tau_m}^{m+1} = Y_{\tau_m}^m, \quad \text{for all } m \in \mathbb{N}, \quad (1.10)$$

where the inequality follows from the fact that all Y^m are càdlàg supermartingales, see the proof of Proposition 1.4.

Step 3: A candidate upper bound (\bar{Y}, \bar{Z}) for the chain $((Y^i, Z^i))_{i \in I}$. We construct a candidate upper bound (\bar{Y}, \bar{Z}) for $((Y^i, Z^i))_{i \in I}$ satisfying $P[\tau(\bar{Y}) > \tau^* \mid \tau^* < T] = 1$, with $\tau(\bar{Y})$ as in (1.9).

To this end, let $(\bar{\sigma}_n)$ be a decreasing sequence of stopping times taking values in the rationals and converging towards τ^* from the right³. Then the stopping times $\hat{\sigma}_n := \bar{\sigma}_n \wedge T$ satisfy $\hat{\sigma}_n > \tau^*$ and $\hat{\sigma}_n \in \mathbb{Q}$, on $\{\tau^* < T\}$, for all n big enough. Let us furthermore define the following stopping time

$$\bar{\tau} := \inf \left\{ t > \tau^* : |\mathcal{E}_{\tau^*}^g(\xi) - \mathcal{E}_t^g(\xi)| > \frac{\varepsilon}{2} \right\} \wedge T. \quad (1.11)$$

Due to the right-continuity of $\mathcal{E}^g(\xi)$ in τ^* , it follows that $\bar{\tau} > \tau^*$ on $\{\tau^* < T\}$. We now set

$$\sigma_n := \hat{\sigma}_n \wedge \bar{\tau}, \quad \text{for all } n \in \mathbb{N}. \quad (1.12)$$

The above stopping times still satisfy $\lim_{n \rightarrow \infty} \sigma_n = \tau^*$ and $\sigma_n > \tau^*$ on $\{\tau^* < T\}$, for all $n \in \mathbb{N}$. We further define the following sets

$$A_n := \left\{ \left| \mathcal{E}_{\tau^*}^g(\xi) - \hat{\mathcal{E}}_{\sigma_n}^g(\xi) \right| < \frac{\varepsilon}{8}, \quad \text{for all } m \geq n \right\} \cap \{\sigma_n \in \mathbb{Q} \cup \{T\}\}. \quad (1.13)$$

They satisfy $A_n \subset A_{n+1}$ and $\bigcup_n A_n = \Omega$, by definition of the sequence (σ_m) ⁴. Note further that $A_n \in \mathcal{F}_{\sigma_n}$, since $\hat{\mathcal{E}}_{\sigma_n}^g(\xi)$ is \mathcal{F}_{σ_n} -measurable, for all $m \geq n$, see Remark 1.3. Since the range of each σ_n is countable on the set A_n , we deduce by Remark 1.3 that,

³ Compare Karatzas and Shreve [45, Problem 2.24].

⁴ Since on $\{\tau^* < T\}$, $\bar{\tau} > \tau^*$ and $\lim_n \hat{\sigma}_n = \tau^*$ with $\hat{\sigma}_n \in \mathbb{Q} \cup \{T\}$, it is ensured that there exists some $n_0 \in \mathbb{N}$, depending on ω , such that σ_n takes values in the rationals for all $n \geq n_0$. By definition of $\mathcal{E}^g(\xi)$ as the right-hand side limit of $\hat{\mathcal{E}}^g(\xi)$ on the rationals, the inequality in the definition of A_n is eventually satisfied for all $m \geq n \geq n_0$.

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for each $n \in \mathbb{N}$, there exists $(\tilde{Y}^n, \tilde{Z}^n) \in \mathcal{A}(\xi, g)$ such that

$$\tilde{Y}_{\sigma_n}^n \leq \hat{\mathcal{E}}_{\sigma_n}^g(\xi) + \frac{\varepsilon}{8} \quad \text{on the set } A_n. \quad (1.14)$$

Next we partition Ω into $B_n := A_n \setminus A_{n-1}$, where we set $A_0 := \emptyset$ and $\tau_0 := 0$, and define the candidate upper bound as

$$\begin{aligned} \bar{Y} = \sum_{m \geq 1} Y^m 1_{[\tau_{m-1}, \tau_m[} \\ + 1_{\{\tau^* < T\}} \sum_{n \geq 1} 1_{B_n} \left[\left(\mathcal{E}_{\tau^*}^g(\xi) + \frac{\varepsilon}{2} \right) 1_{[\tau^*, \sigma_n[} + \tilde{Y}^n 1_{[\sigma_n, T[} \right], \quad \bar{Y}_T = \xi \end{aligned} \quad (1.15)$$

$$\bar{Z} = \sum_{m \geq 1} Z^m 1_{] \tau_{m-1}, \tau_m]} + 1_{\{\tau^* < T\}} \sum_{n \geq 1} \tilde{Z}^n 1_{B_n} 1_{] \sigma_n, T]}. \quad (1.16)$$

Step 4: Verification of $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$. By verifying that the pair (\bar{Y}, \bar{Z}) is an element of $\mathcal{A}(\xi, g)$, we identify (\bar{Y}, \bar{Z}) as an upper bound for the chain $((Y^i, Z^i))_{i \in I}$. Even more, $P[\tau(\bar{Y}) > \tau^* \mid \tau^* < T] = 1$ holds true, since, on the set B_n , we have $\bar{Y}_t = \mathcal{E}_{\tau^*}^g(\xi) + \frac{\varepsilon}{2} \leq \mathcal{E}_t^g(\xi) + \varepsilon$, for all $t \in [\tau^*, \sigma_n[$, due to the definition of $\bar{\tau}$ in (1.11).

Step 4a: The value process \bar{Y} is an element of \mathcal{S} . By construction, the only thing to show is that \bar{Y}_{τ^*-} , the left limit at τ^* , exists. This follows from Proposition 1.4, since, by means of $((Y^m, Z^m)) \subset \mathcal{A}(\xi, g)$ and $\xi \in L^1(\mathcal{F}_T)$, all Y^m are càdlàg supermartingales, see Lemma 1.1, which are bounded from below by a uniformly integrable martingale, more precisely $Y^m \geq E[\xi \mid \mathcal{F}]$, for all $m \in \mathbb{N}$, and satisfy (1.10).

Step 4b: The control process \bar{Z} is an element of \mathcal{L} and admissible. We proceed by defining, for each $n \in \mathbb{N}$, the processes $\bar{Z}^n := \sum_{m=1}^n Z^m 1_{] \tau_{m-1}, \tau_m]} = \bar{Z} 1_{[0, \tau_n]} = Z^n 1_{[0, \tau_n]}$ and $N^n := \int \bar{Z}^n dW = \int Z^n 1_{[0, \tau_n]} dW$, where the equalities follow from (1.8). Observe that $N^{n+1} 1_{[0, \tau_n]} = N^n 1_{[0, \tau_n]}$, for all $n \in \mathbb{N}$, and that (POS), (1.1) and the supermartingale property of $\int Z^n dW$ imply

$$N^n 1_{[\tau_{n-1}, \tau_n[} \geq 1_{[\tau_{n-1}, \tau_n[} (-E[\xi^- \mid \mathcal{F}] - Y_0^b). \quad (1.17)$$

By means of (1.17) and since $\xi^- \in L^1(\mathcal{F}_T)$, with $N^\infty := \lim_n N_{\tau_{n-1}}^n$, the process

$$N = \sum_{n \geq 1} N^n 1_{[\tau_{n-1}, \tau_n[} + 1_{[\tau^*, T]} N^\infty$$

is a well-defined continuous supermartingale due to Proposition 1.4. Hence we may define a localizing sequence by setting $\kappa_n := \inf\{t \geq 0 : |N_t| > n\} \wedge T$ and deduce that N is a continuous local martingale, because N^{κ_n} is a uniformly integrable martingale, for all $n \in \mathbb{N}$. Indeed, for each $n \in \mathbb{N}$ and $m \in \mathbb{N}$, the process $(N^m)^{\kappa_n}$, being a bounded stochastic integral, is a martingale. Moreover, the family $(N_{\kappa_n \wedge t}^m)_{m \in \mathbb{N}}$ is uniformly integrable and $N_{\kappa_n \wedge t} = \lim_m N_{\kappa_n \wedge t}^m$, for all $t \in [0, T]$. Consequently, $E[N_t^{\kappa_n} \mid \mathcal{F}_s] =$

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$\lim_m E[N_{\kappa_n \wedge t}^m | \mathcal{F}_s] = \lim_m N_{\kappa_n \wedge s}^m = N_s^{\kappa_n}$, for all $0 \leq s \leq t \leq T$, and the claim follows. Since the quadratic variation of a continuous local martingale is continuous and unique, see Karatzas and Shreve [45, page 36], we obtain

$$\int_0^{\tau^*} \bar{Z}_u^2 du = \lim_{n \rightarrow \infty} \int_0^{\kappa_n \wedge \tau^*} \bar{Z}_u^2 du = \lim_{n \rightarrow \infty} \langle N \rangle_{\kappa_n \wedge \tau^*} = \langle N \rangle_{\tau^*} < \infty.$$

Observe that $\sigma := \sum_{n \geq 1} 1_{B_n} \sigma_n$ is an element of \mathcal{T} . Indeed, $\{\sigma \leq t\} = \bigcup_{n \geq 1} (B_n \cap \{\sigma_n \leq t\}) \in \mathcal{F}_t$, for all $t \in [0, T]$, since $B_n \in \mathcal{F}_{\sigma_n}$. From $\bar{Z}1_{[\tau^*, \sigma]} = 0$ we get that

$$\int_0^T \bar{Z}_u^2 du = \langle N \rangle_{\tau^*} + 1_{\{\tau^* < T\}} \sum_{n \geq 1} 1_{B_n} \int_{\sigma}^T (\tilde{Z}_u^n)^2 du < \infty,$$

since $(\tilde{Z}^n) \subset \mathcal{L}$. Hence we conclude that $\bar{Z} \in \mathcal{L}$. As for the supermartingale property of $\int \bar{Z} dW$, observe that

$$\begin{aligned} \int_0^{t \wedge \tau^*} \bar{Z}_u dW_u &= \lim_{n \rightarrow \infty} \int_0^{t \wedge \tau_n} Z_u^n dW_u \\ &\geq \lim_{n \rightarrow \infty} -E[\xi^- | \mathcal{F}_{t \wedge \tau_n}] - Y_0^b = -E[\xi^- | \mathcal{F}_{t \wedge \tau^*}] - Y_0^b, \end{aligned}$$

where the inequality follows from (1.1) and (POS). Being bounded from below by a martingale, we deduce by Fatou's lemma that $\bar{Z}1_{[0, \tau^*]}$ is admissible. Since $\bar{Z}1_{[\tau^*, \sigma]} = 0$ and all \tilde{Z}^n are admissible, it follows from Proposition 1.2 that \bar{Z} is indeed admissible.

Step 4c: The pair (\bar{Y}, \bar{Z}) is a supersolution. Finally, showing that (\bar{Y}, \bar{Z}) satisfies (1.1) identifies (\bar{Y}, \bar{Z}) as an element of $\mathcal{A}(\xi, g)$. Observe first that, for all $0 \leq s \leq t \leq T$ and all $m \in \mathbb{N}$, the expression $\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^t \bar{Z}_u dW_u$ can be written as

$$\begin{aligned} \bar{Y}_s &- \int_s^{(\tau_m \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^{(\tau_m \vee s) \wedge t} \bar{Z}_u dW_u \\ &- \int_s^{(\tau^* \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^{(\tau^* \vee s) \wedge t} \bar{Z}_u dW_u - \int_s^{(\sigma \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du \\ &\quad + \int_s^{(\sigma \vee s) \wedge t} \bar{Z}_u dW_u - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^t \bar{Z}_u dW_u. \quad (1.18) \end{aligned}$$

Now, we have that

$$\bar{Y}_s - \int_s^{(\tau_m \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^{(\tau_m \vee s) \wedge t} \bar{Z}_u dW_u \geq \bar{Y}_{(\tau_m \vee s) \wedge t}, \quad (1.19)$$

by Proposition 1.2, since $((Y^m, Z^m)) \subset \mathcal{A}(\xi, g)$ and $Y_{\tau_m-}^m \geq Y_{\tau_m}^{m+1}$, for all $m \in \mathbb{N}$, due to (1.10). By letting m tend to infinity and noting that

$$\lim_{m \rightarrow \infty} \int_{(\tau_m \vee s) \wedge t}^{(\tau^* \vee s) \wedge t} \bar{Z}_u dW_u = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \int_{(\tau_m \vee s) \wedge t}^{(\tau^* \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du = 0,$$

(1.18) and (1.19) yield that

$$\begin{aligned} \bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^t \bar{Z}_u dW_u \\ \geq \bar{Y}_{((\tau^*-) \vee s) \wedge t} - \int_{(\tau^* \vee s) \wedge t}^{(\sigma \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du + \int_{(\tau^* \vee s) \wedge t}^{(\sigma \vee s) \wedge t} \bar{Z}_u dW_u \\ - \int_{(\sigma \vee s) \wedge t}^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_{(\sigma \vee s) \wedge t}^t \bar{Z}_u dW_u. \end{aligned} \quad (1.20)$$

We now use that \bar{Y} can only jump downwards at τ^* . Indeed, since \bar{Y} is càdlàg, in particular \bar{Y}_{τ^*-} , the left limit at τ^* , exists and is unique, P -almost surely. Furthermore, it holds that $\lim_{m \rightarrow \infty} \bar{Y}_{\tau_m-} = \bar{Y}_{\tau^*-}$. Indeed, since the left limits \bar{Y}_{τ_m-} are well-defined, for all $m \in \mathbb{N}$, we can choose a sequence of stopping times (η_m) such that $\eta_m \in [\tau_{m-1}, \tau_m[$ and $|\bar{Y}_{\tau_m-} - \bar{Y}_{\eta_m}| < \frac{1}{m}$. Since $\lim_m \eta_m = \tau^*$ and \bar{Y} is càdlàg, in particular holds $\lim_m \bar{Y}_{\eta_m} = \bar{Y}_{\tau^*-}$ and the claim follows by an application of the triangular inequality. Thus

$$\begin{aligned} \bar{Y}_{\tau^*-} &= \lim_{m \rightarrow \infty} \bar{Y}_{\tau_m-} = \lim_{m \rightarrow \infty} Y_{\tau_m-}^m \geq \lim_{m \rightarrow \infty} Y_{\tau_m}^m \\ &\geq \lim_{m \rightarrow \infty} \mathcal{E}_{\tau_m}^g(\xi) + \varepsilon = \mathcal{E}_{\tau^*-}^g(\xi) + \varepsilon \geq \mathcal{E}_{\tau^*}^g(\xi) + \varepsilon > \bar{Y}_{\tau^*}. \end{aligned}$$

The first and third inequality hold, since a càdlàg supermartingale can only jump

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downwards, see the proof of Proposition 1.4. Hence, (1.20) can be further estimated by

$$\begin{aligned} \bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^t \bar{Z}_u dW_u \\ \geq \bar{Y}_{(\tau^* \vee s) \wedge t} - \int_{(\sigma \vee s) \wedge t}^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_{(\sigma \vee s) \wedge t}^t \bar{Z}_u dW_u, \end{aligned}$$

where we used that

$$\int_{(\tau^* \vee s) \wedge t}^{(\sigma \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du = \int_{(\tau^* \vee s) \wedge t}^{(\sigma \vee s) \wedge t} \bar{Z}_u dW_u = 0,$$

due to (1.16), the definition of σ , and (NOR). Now observe that $\bar{Y}_{(\tau^* \vee s) \wedge t} \geq \bar{Y}_{(\sigma \vee s) \wedge t}$, since $\bar{Y}1_{[\tau^*, \sigma[} = (\mathcal{E}_{\tau^*}^g(\xi) + \frac{\varepsilon}{2})1_{[\tau^*, \sigma[}$ and \bar{Y} can only jump downwards at σ . Indeed, on the set B_n , by means of (1.15), (1.13), and (1.14) holds

$$\begin{aligned} \bar{Y}_{\sigma_n-} &= \mathcal{E}_{\tau^*}^g(\xi) + \frac{\varepsilon}{2} = \mathcal{E}_{\tau^*}^g(\xi) - \hat{\mathcal{E}}_{\sigma_n}^g(\xi) + \hat{\mathcal{E}}_{\sigma_n}^g(\xi) + \frac{\varepsilon}{2} \\ &\geq -\frac{\varepsilon}{8} + \hat{\mathcal{E}}_{\sigma_n}^g(\xi) + \frac{\varepsilon}{2} \geq \tilde{Y}_{\sigma_n}^n - \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \tilde{Y}_{\sigma_n}^n = \bar{Y}_{\sigma_n}. \end{aligned}$$

Consequently,

$$\begin{aligned} \bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^t \bar{Z}_u dW_u \\ \geq \bar{Y}_{(\sigma \vee s) \wedge t} - \int_{(\sigma \vee s) \wedge t}^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_{(\sigma \vee s) \wedge t}^t \bar{Z}_u dW_u \geq \bar{Y}_t, \quad (1.21) \end{aligned}$$

where the second inequality in (1.21) follows from $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{A}(\xi, g)$ and Proposition 1.2.

Step 5: The maximal element (Y^M, Z^M) . By Zorn's lemma, there exists a maximal element (Y^M, Z^M) in $\mathcal{A}(\xi, g)$ with respect to the preorder (1.8), satisfying, without loss of generality, $Y_T^M = \xi$. Finally, by showing that the corresponding stopping time satisfies $\tau^M = T$, we obtain a supersolution (Y^M, Z^M) satisfying $\|Y^M - \mathcal{E}^g(\xi)\|_{\mathcal{R}^\infty} \leq \varepsilon$, due to the definition of τ^M in analogy to (1.9). Thus, choosing $Y^M = Y^\varepsilon$ in (1.7) finishes our proof.

But on $\{\tau^M < T\}$ we consider the chain consisting only of (Y^M, Z^M) and, analogously to (1.15) and (1.16), construct an upper bound (\bar{Y}, \bar{Z}) , with corresponding stopping time $\tau(\bar{Y})$ as in (1.9), satisfying $P[\tau(\bar{Y}) > \tau^M \mid \tau^M < T] = 1$. This yields $P[\tau^M < T] \leq$

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$P[\tau(\bar{Y}) > \tau^M] = 0$, due to the maximality of τ^M . Hence we deduce that $\tau^M = T$. \square

The techniques used in the proof of Theorem 1.5 show that $\mathcal{A}(\xi, g)$ exhibits a certain closedness under monotone limits of decreasing supersolutions.

Theorem 1.6. *Let g be a generator satisfying (LSC), (POS) and (NOR) and $\xi \in L^0(\mathcal{F}_T)$ a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. Let furthermore $((Y^n, Z^n))$ be a decreasing sequence within $\mathcal{A}(\xi, g)$ with pointwise limit $\hat{Y}_t := \lim_{n \rightarrow \infty} Y_t^n$, for $t \in [0, T]$. Then \hat{Y} is a supermartingale and it holds*

$$\hat{Y}_t \geq Y_t := \lim_{s \downarrow t, s \in \mathbb{Q}} \hat{Y}_s, \quad \text{for all } t \in [0, T].$$

Furthermore, with $Y_T := \xi$, there is a sequence $((\tilde{Y}^n, \tilde{Z}^n))$ within $\mathcal{A}(\xi, g)$ such that $\lim_{n \rightarrow \infty} \|\tilde{Y}^n - Y\|_{\mathcal{R}^\infty} = 0$, and a unique control $Z \in \mathcal{L}$ such that $(Y, Z) \in \mathcal{A}(\xi, g)$.

Proof. The proof is a straightforward adaptation of the proof of Theorem 1.5. \square

Now we focus on the question whether it is possible to find a minimal supersolution within $\mathcal{A}(\xi, g)$, the associated control process Z of which belongs to \mathcal{L}^1 , and $\int Z dW$ therefore constitutes a true martingale instead of only a supermartingale. To this end, we consider the following subset of $\mathcal{A}(\xi, g)$

$$\mathcal{A}^1(\xi, g) := \{(Y, Z) \in \mathcal{A}(\xi, g) : Z \in \mathcal{L}^1\}.$$

By imposing stronger assumptions on the terminal condition ξ , the next theorem yields the existence of a unique minimal supersolution in $\mathcal{A}^1(\xi, g)$.

Theorem 1.7. *Assume that the generator g satisfies (LSC), (POS) and (NOR), and let $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $(E[\xi^- | \mathcal{F}])^* \in L^1(\mathcal{F}_T)$. If $\mathcal{A}^1(\xi, g) \neq \emptyset$, then there exists a control \hat{Z} such that $(\mathcal{E}^g(\xi), \hat{Z})$ is the unique minimal supersolution in $\mathcal{A}^1(\xi, g)$.*

Proof. $\mathcal{A}^1(\xi, g) \neq \emptyset$ yields that $\mathcal{A}(\xi, g) \neq \emptyset$, because $\mathcal{A}^1(\xi, g) \subseteq \mathcal{A}(\xi, g)$. Also, from $(E[\xi^- | \mathcal{F}])^*_T \in L^1(\mathcal{F}_T)$ we deduce that $\xi^- \in L^1(\mathcal{F}_T)$. Hence, Theorem 1.5 yields the existence of a unique control \hat{Z} such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. Verifying that $\hat{Z} \in \mathcal{L}^1$ is done as in Drapeau et al. [24, Theorem 4.5]. \square

1.3.3. Relaxations of the Conditions (NOR) and (POS)

In this section, we discuss possible relaxations of the conditions (NOR) and (POS) imposed on the generator throughout Sections 1.3.1 and 1.3.2.

First, we want to replace (NOR) by the weaker assumption (NOR'). We say that a generator g satisfies

(NOR') if, for all $\tau \in \mathcal{T}$, there exists some stopping time $\delta > \tau$ such that the stochastic differential equation

$$dy_s = -g_s(y_s, 0)ds, \quad y_\tau = \mathcal{E}_\tau^g(\xi) + \frac{\varepsilon}{2} \quad (1.22)$$

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admits a solution on $[\tau, \delta]$ where we set $g_t(y, 0) = 0$, for all $y \in \mathbb{R}$ and $t > T$.

Remark 1.8. It is possible to relax the condition (NOR') further by requiring that the stochastic differential inequality $dy_s \geq -g_s(y_s, 0)ds$ with initial value $y_\tau = \mathcal{E}_\tau^g(\xi) + \frac{\varepsilon}{2}$ has a càdlàg solution y on $[\tau, \delta)$. \blacklozenge

By this we obtain the following extension of Theorem 1.5.

Theorem 1.9. *Let g be a generator satisfying (LSC), (POS) and (NOR') and $\xi \in L^0(\mathcal{F}_T)$ a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. If $\mathcal{A}(\xi, g) \neq \emptyset$, then there exists a unique control process $\hat{Z} \in \mathcal{L}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$.*

Proof. The proof is almost the same as the proof of Theorem 1.5. The only difference lies in the definition of \bar{Y} in (1.15). After τ^* , instead of extending by a constant function, we concatenate the value process at τ^* with the solution of the SDE (1.22), started at $y_{\tau^*} = \mathcal{E}_{\tau^*}^g(\xi) + \frac{\varepsilon}{2}$ and denoted by y . We emphasize that the zero control is maintained. We only need to adjust the argumentation in Step 4c. To that end, we introduce the stopping time

$$\kappa := \inf \left\{ t > \tau^* : \int_{\tau^*}^t g_s(y_s, 0)ds > \frac{\varepsilon}{8} \right\} \wedge \delta, \quad (1.23)$$

and use $\bar{\kappa} := \kappa \wedge \bar{\tau}$, with $\bar{\tau}$ as in (1.11), within the definition of the sequence (σ_n) in analogy to (1.12), that is, $\sigma_n = \hat{\sigma}_n \wedge \bar{\kappa}$, for all $n \in \mathbb{N}$. As before, we set $\sigma := \sum_{n \geq 1} 1_{B_n} \sigma_n$. Consequently, \bar{Y} is given by

$$\begin{aligned} \bar{Y} = \sum_{m \geq 1} Y^m 1_{[\tau_{m-1}, \tau_m[} + 1_{\{\tau^* < T\}} \sum_{n \geq 1} 1_{B_n} \left(\mathcal{E}_{\tau^*}^g(\xi) + \frac{\varepsilon}{2} - \int_{\tau^*}^{\cdot} g_s(y_s, 0)ds \right) 1_{[\tau^*, \sigma_n[} \\ + 1_{\{\tau^* < T\}} \sum_{n \geq 1} 1_{B_n} \bar{Y}^n 1_{[\sigma_n, T[} \quad , \quad \bar{Y}_T = \xi. \end{aligned}$$

The definition of the stopping time $\bar{\tau}$ implies that, on the set B_n , we have $\bar{Y}_t \leq \mathcal{E}_t^g(\xi) + \varepsilon$, for all $t \in [\tau^*, \sigma_n[$. Indeed, observe that, for $t \in [\tau^*, \sigma_n[$,

$$\bar{Y}_t = \mathcal{E}_{\tau^*}^g(\xi) + \frac{\varepsilon}{2} - \int_{\tau^*}^t g_s(y_s, 0)ds \leq \mathcal{E}_t^g(\xi) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \mathcal{E}_t^g(\xi) + \varepsilon.$$

Furthermore, on the set B_n , by means of (1.13), (1.23), and (1.14),

$$\begin{aligned}
 \bar{Y}_{\sigma_n-} &= \mathcal{E}_{\tau^*}^g(\xi) + \frac{\varepsilon}{2} - \int_{\tau^*}^{\sigma_n} g_s(y_s, 0) ds \\
 &= \mathcal{E}_{\tau^*}^g(\xi) - \hat{\mathcal{E}}_{\sigma_n}^g(\xi) + \hat{\mathcal{E}}_{\sigma_n}^g(\xi) + \frac{\varepsilon}{2} - \int_{\tau^*}^{\sigma_n} g_s(y_s, 0) ds \\
 &\geq \frac{3\varepsilon}{8} + \hat{\mathcal{E}}_{\sigma_n}^g(\xi) - \int_{\tau^*}^{\sigma_n} g_s(y_s, 0) ds \geq \frac{2\varepsilon}{8} + \hat{\mathcal{E}}_{\sigma_n}^g(\xi) \geq \bar{Y}_{\sigma_n}.
 \end{aligned}$$

Hence, pasting at the stopping time σ is in accordance with Proposition 1.2. This yields the result. \square

As in Drapeau et al. [24], the positivity assumption (POS) on the generator can be relaxed to a linear lower bound, which, however, has to be consistent with the assumption (NOR'). In the following a generator g is said to be

(LB-NOR') linearly bounded from below under (NOR') if there exist adapted measurable processes a and b with values in $\mathbb{R}^{1 \times d}$ and \mathbb{R} , respectively, such that $g(y, z) \geq az^T - b$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$, and

$$\frac{dP^a}{dP} = \mathcal{E} \left(\int adW \right)_T \quad (1.24)$$

defines an equivalent probability measure P^a . Furthermore, $\int_0^t b_s ds \in L^1(P^a)$ holds for all $t \in [0, T]$, and a and b are such that the positive generator defined by

$$\bar{g}(y, z) := g \left(y + \int_0^\cdot b_s ds, z \right) - az^T - b, \text{ for all } (y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}, \quad (1.25)$$

satisfies (NOR').

An (LB-NOR') setting can always be reduced to a setting with generator satisfying (POS) and (NOR'), by using the change of measure (1.24) and \bar{g} defined in (1.25). Hence, Lemma 1.1 and Proposition 1.2, which strongly rely on the property (POS), can be applied. However, we need a slightly different definition of admissibility than before. A control process Z is said to be a -admissible if $\int Z dW^a$ is a P^a -supermartingale, where $W^a = W - \int a ds$ is a P^a -Brownian motion by Girsanov's theorem. The set $\mathcal{A}^a(\xi, g) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : Z \text{ is } a\text{-admissible and (1.1) holds}\}$, as well as the process

$$\hat{\mathcal{E}}_t^{g,a}(\xi) = \text{ess inf} \{Y_t \in L^0(\mathcal{F}_t) : (Y, Z) \in \mathcal{A}^a(\xi, g)\}, \quad \text{for } t \in [0, T],$$

are defined analogously to (1.3) and (1.4), respectively. We are now ready to state our

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most general result, which follows from Theorem 1.9 and Drapeau et al. [24, Theorem 4.17].

Theorem 1.10. *Let g be a generator satisfying (LSC) and (LB-NOR') and $\xi \in L^0(\mathcal{F}_T)$ a terminal condition such that $\xi^- \in L^1(P^a)$. If in addition $\mathcal{A}^a(\xi, g) \neq \emptyset$, then*

$$\mathcal{E}_t^{g,a}(\xi) := \lim_{s \downarrow t, s \in \mathbb{Q}} \hat{\mathcal{E}}_s^{g,a}(\xi), \quad \text{for all } t \in [0, T) \quad \text{and} \quad \mathcal{E}_T^{g,a}(\xi) := \xi$$

is the value process of the unique minimal supersolution, that is, there exists a unique control process \hat{Z} such that $(\mathcal{E}^{g,a}(\xi), \hat{Z}) \in \mathcal{A}^a(\xi, g)$.

1.3.4. Continuous Local Martingales and Controls in \mathcal{L}^1

Under stronger integrability conditions, the techniques used in the proof of Theorem 1.5 can be generalized to the case where the Brownian motion W appearing in the stochastic integral in (1.1) is replaced by a d -dimensional continuous local martingale M . Let us assume that M is adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$, which satisfies the usual conditions and in which all martingales are continuous and all stopping times are predictable. We consider controls within the set $\mathcal{L}^1 := \mathcal{L}^1(M)$, consisting of all $\mathbb{R}^{1 \times d}$ -valued, progressively measurable processes Z , such that $\int Z dM \in \mathcal{H}^1$. As before, for $Z \in \mathcal{L}^1$ the stochastic integral $\int Z dM$ is well defined and is by means of the Burkholder-Davis-Gundy inequality a continuous martingale. A pair $(Y, Z) \in \mathcal{S} \times \mathcal{L}^1$ is now called a supersolution of a BSDE if it satisfies, for $0 \leq s \leq t \leq T$,

$$Y_s - \int_s^t g_u(Y_u, Z_u) d\langle M \rangle_u + \int_s^t Z_u dM_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi, \quad (1.26)$$

for a generator g and a terminal condition $\xi \in L^0(\mathcal{F}_T)$. We will focus on the set

$$\mathcal{A}^{M,1}(\xi, g) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L}^1 : (Y, Z) \text{ satisfy (1.26)}\}.$$

If we assume $\mathcal{A}^{M,1}(\xi, g)$ to be non-empty, Theorem 1.5 combined with compactness results for sequences of \mathcal{H}^1 -bounded martingales given in Delbaen and Schachermayer [19] yields that

$$\mathcal{E}_t^g(\xi) := \lim_{s \downarrow t, s \in \mathbb{Q}} \hat{\mathcal{E}}_s^g(\xi), \quad \text{for all } t \in [0, T) \quad \text{and} \quad \mathcal{E}_T^g(\xi) := \xi,$$

where

$$\hat{\mathcal{E}}_t^g(\xi) := \text{ess inf} \{Y_t \in L^0(\mathcal{F}_t) : (Y, Z) \in \mathcal{A}^{M,1}(\xi, g)\}, \quad t \in [0, T],$$

is the value process of the unique minimal supersolution within $\mathcal{A}^{M,1}(\xi, g)$. Note that Lemma 1.1 and Proposition 1.2 extend to the case where W is substituted by M .

Theorem 1.11. *Assume that the generator g satisfies (LSC), (POS) and (NOR) and let $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $(E[\xi^- | \mathcal{F}_\cdot])^* \in L^1(\mathcal{F}_T)$. If $\mathcal{A}^{M,1}(\xi, g) \neq \emptyset$, then there exists a unique control \hat{Z} such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}^{M,1}(\xi, g)$.*

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Proof. By assumption, there is some $(Y^b, Z^b) \in \mathcal{A}^{M,1}(\xi, g)$ and we consider, without loss of generality, only those pairs $(Y, Z) \in \mathcal{A}^{M,1}(\xi, g)$ satisfying $Y \leq Y^b$, obtained by suitable pasting as in Proposition 1.2. Using the techniques of the proof of Theorem 1.5, we can find a sequence $((Y^n, Z^n)) \subset \mathcal{A}^{M,1}(\xi, g)$ satisfying $\lim_n \|Y^n - \mathcal{E}^g(\xi)\|_{\mathcal{R}^\infty} = 0$, in analogy to (1.6). Since $(\int Z^n dM)$ is uniformly bounded in \mathcal{H}^1 , compare Drapeau et al. [24, Theorem 4.5], it follows from Barlow and Protter [4, Theorem 1] that $\mathcal{E}^g(\xi)$ is a special semimartingale with canonical decomposition $\mathcal{E}^g(\xi) = \mathcal{E}_0^g(\xi) + N - A$ and that

$$\lim_{n \rightarrow \infty} \left\| \int Z^n dM - N \right\|_{\mathcal{H}^1} = 0. \quad (1.27)$$

Moreover, $N \in \mathcal{H}^1$. Now Delbaen and Schachermayer [19, Theorem 1.6] yields the existence of some $\hat{Z} \in \mathcal{L}^1$ such that $N = \int \hat{Z} dM$. By means of (1.27), (Z^n) converges, up to a subsequence, $P \otimes d\langle M \rangle_t$ -almost everywhere to \hat{Z} and $\lim_n \int_0^t Z^n dM = \int_0^t \hat{Z} dM$, for all $t \in [0, T]$, P -almost surely, by means of the Burkholder-Davis-Gundy inequality. In particular, $\lim_{n \rightarrow \infty} Z^n(\omega) = \hat{Z}(\omega)$, $d\langle M \rangle$ -almost everywhere, for almost all $\omega \in \Omega$. Verifying that $(\mathcal{E}^g(\xi), \hat{Z})$ satisfy (1.26) is now done analogously to Step 1 in the proof of Theorem 1.5, and hence we are done. \square

2. Minimal Supersolutions of Convex BSDEs under Constraints – Existence and Duality

2.1. Introduction

Opposed to the first chapter where we worked in an unconstrained framework, it is the principal aim of the current chapter to add additional constraints on the set of admissible control processes within the framework of supersolutions of BSDEs. More specifically, on a filtered probability space, the filtration of which is generated by a d -dimensional Brownian motion, we are interested in quadruplets (Y, Z, Δ, Γ) of processes such that, for all $0 \leq s \leq t \leq T$, the system

$$\begin{aligned} Y_s - \int_s^t g_u(Y_u, Z_u, \Delta_u, \Gamma_u) du + \int_s^t Z_u dW_u &\geq Y_t, \quad Y_T \geq \xi, \\ Z_t &= z + \int_0^t \Delta_u du + \int_0^t \Gamma_u dW_u \end{aligned} \quad (2.1)$$

is satisfied. Here, for ξ a terminal condition, Y is the càdlàg value process and Z the continuous control process with decomposition (Δ, Γ) . The generator g is assumed to be jointly convex and may depend on the decomposition of the continuous semimartingale Z which accounts for the expression “delta and gamma constraints”. It is our objective to give conditions ensuring that the set $\mathcal{A}(\xi, g)$, consisting of all admissible pairs (Y, Z) satisfying (2.1), contains elements (\hat{Y}, \hat{Z}) that are minimal at finitely many times within $[0, T]$ in the following sense: The value process \hat{Y} of a minimal supersolution is less or equal than all value processes of supersolutions within the class where controls coincide up to these times. Furthermore, we derive stability properties of the non-linear operator $\mathcal{E}_0^g(\cdot, z)$ that maps a terminal condition to the value of the minimal supersolution at time zero, enabling us to study the dual problem. Finally, we give conditions relying on the aforementioned BSDE duality for the existence of solutions under constraints.

Finding the minimal initial value of a supersolution under constraints is closely related to the superreplication problem in a financial market under gamma constraints, first studied in Soner and Touzi [63]. Indeed, the classical gamma constraints can be incorporated into our more general framework by setting the generator to $+\infty$ whenever the diffusion part Γ is outside a predetermined interval. Note that once we have

solved our problem at a single time, it is rather straightforward to derive the existence of supersolutions minimal at finitely many times.

In a nutshell, inspired by the methods first used in Drapeau et al. [24] and then later in Heyne et al. [38], we consider the operator $\mathcal{E}_t^g(\xi, V) := \text{ess inf}\{Y_t : (Y, Z) \in \mathcal{A}(\xi, g) \text{ and } Z_t = V\}$ where V is a random variable attainable by controls at time t . The structure of $\mathcal{E}_t^g(\xi, V)$ is owed to the following consideration. In the case of constraints on the controls, stability of supersolutions with respect to concatenating, a property extensively used in Drapeau et al. [24] and Heyne et al. [38], is preserved only subject to equality of the respective control processes at times of pasting. We show that, for each fixed time $t \in [0, T]$, the set of supersolutions (Y, Z) satisfying $Y_t = \mathcal{E}_t^g(\xi, Z_t)$ is non-empty. In order to do so, we impose a superquadratic growth condition in the decomposition parts (Δ, Γ) of controls on the generator g , reflecting a penalization of rapid changes in control values. The consequence is twofold. First, it ensures that the sequence of stochastic integrals $(\int Z^n dW)$ corresponding to the minimizing sequence $Y_t^n \downarrow \mathcal{E}_t^g(\xi, Z_t)$ is bounded in \mathcal{H}^2 . Drawing from compactness results for the space of martingales \mathcal{H}^2 given in Delbaen and Schachermayer [19], we obtain our candidate control process \hat{Z} as the limit of a sequence in the asymptotic convex hull of (Z^n) . At this point it is crucial to preserve the continuous semimartingale structure of the limit object, possible by using once more the aforementioned growth condition on g .

Finally, we provide stability results of the operator $\mathcal{E}_0^g(\cdot, z)$ at time zero such as monotone convergence, Fatou's lemma and L^1 -lower semicontinuity. This, together with convexity, gives way to a dual representation of \mathcal{E}_0^g as a consequence of the Fenchel-Moreau theorem. We characterize the conjugate \mathcal{E}_0^* in terms of the decomposition parts of the controls and show that \mathcal{E}_0^* is always attained. In particular, for the case of a quadratic generator we show that it is possible to explicitly compute the conjugate by means of classical calculus of variations methods, giving additional structural insight into the problem. If we assume in turn the existence of an optimal subgradient such that $\mathcal{E}_0^g(\xi, z)$ is attained in its dual representation, we can prove that the associated BSDE with parameters (ξ, g) admits a solution under constraints. This extends results of Delbaen et al. [22] and Drapeau et al. [25] obtained in the unconstrained case.

Before we continue, let us briefly discuss the existing literature on the subject. Ever since the seminal paper Pardoux and Peng [54], an extensive amount of work has been done in the field of BSDEs, resulting in such important contributions as for instance El Karoui et al. [30], Kobylanski [52] or Briand and Hu [8]. We refer the reader to Peng [56] or Drapeau et al. [24] for a more thorough treatment of the literature concerning solutions and in particular supersolutions of BSDEs. There are many works dealing with optimization or (super-)replication under constraints, see for instance Cvitanic and Karatzas [17], Jouini and Kallal [44] or Broadie et al. [9] and references therein, but the notion of gamma constraints in the context of superhedging was introduced in Soner and Touzi [63]. Therein, the authors identify the superreplication cost as the solution to a variational inequality. In Cheridito et al. [14], the aforementioned problem is solved in a multi-dimensional setting and the superreplication price is characterized as the unique viscosity solution of a nonstandard partial differential equation, whereas

in Cheridito et al. [15] the authors treat the related system of BSDEs and SDEs in a more abstract fashion. Compare also the more recent work Soner et al. [64] for a dual characterization of the superreplication problem.

The chapter is organized as follows. We mostly work within the setting introduced in Section 1.2. A precise definition of supersolutions under gamma and delta constraints and minimality in this framework as well as existence and stability results are then given in Section 2.2. Finally, Section 2.3 provides duality results along with explicit computations for a particular generator, and links the duality to the existence of solutions under constraints.

2.2. Minimal Supersolutions of BSDEs under Delta and Gamma Constraints

2.2.1. Definitions

Concerning the notation and conventions in this chapter, we will work within the setting given in Section 1.2. Note in addition that in the following, for $m, n \in \{1, d\}$, the expression $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^{m \times n}$, that is $|x| = (\sum_{ij} x_{ij}^2)^{\frac{1}{2}}$. For a given sequence (x_n) in some convex set, we say that a sequence (\tilde{x}_n) is in the asymptotic convex hull of (x_n) if $\tilde{x}_n \in \text{conv}\{x_n, x_{n+1}, \dots\}$, for all $n \in \mathbb{N}$.

Throughout this chapter, a generator is a jointly measurable function g from $\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{d \times d}$ to $\mathbb{R} \cup \{+\infty\}$ where $\Omega \times [0, T]$ is endowed with the progressive σ -field. A control $Z \in \mathcal{L}$ with initial value $z \in \mathbb{R}^{1 \times d}$ is said to have the decomposition (Δ, Γ) if it is of the form $Z = z + \int \Delta du + \int \Gamma dW$, for progressively measurable (Δ, Γ) taking values in $\mathbb{R}^{1 \times d} \times \mathbb{R}^{d \times d}$.¹ A control is said to be admissible if the continuous local martingale $\int Z dW$ is a supermartingale. Let us collect all these processes in the set Θ defined by

$$\Theta := \left\{ Z \in \mathcal{L} : \begin{array}{l} \exists z \in \mathbb{R}^{1 \times d}, \exists (\Delta, \Gamma) \text{ progressively measurable such that} \\ Z = z + \int \Delta du + \int \Gamma dW \text{ and } \int Z dW \text{ is a supermartingale} \end{array} \right\}.$$

Whenever we want to stress the dependence of controls on a fixed initial value $z \in \mathbb{R}^{1 \times d}$, we make use of the set $\Theta(z) := \{Z \in \Theta : Z_0 = z\}$. Given a generator g and a terminal condition $\xi \in L^0(\mathcal{F}_T)$, a pair $(Y, Z) \in \mathcal{S} \times \Theta$ is a supersolution of a BSDE under gamma and delta constraints if, for $0 \leq s \leq t \leq T$, it holds

$$Y_s - \int_s^t g_u(Y_u, Z_u, \Delta_u, \Gamma_u) du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi. \quad (2.2)$$

¹ In order to be compatible with the dimension of Z , actually the transpose $(\int \Gamma dW)^T$ of $\int \Gamma dW$ needs to be considered. However, we suppress this operation for the remainder in order to keep the notation simple.

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For a supersolution (Y, Z) , we call Y the value process and Z its corresponding control process. We are now interested in the set

$$\mathcal{A}(\xi, g) := \{(Y, Z) \in \mathcal{S} \times \Theta : (2.2) \text{ holds}\} \quad (2.3)$$

and, for $z \in \mathbb{R}^{1 \times d}$, we define the set $\mathcal{A}(\xi, g, z)$ by

$$\mathcal{A}(\xi, g, z) := \{(Y, Z) \in \mathcal{A}(\xi, g) : Z_0 = z\}.$$

Throughout this chapter a generator g is said to be

(LSC) if $(y, z, \delta, \gamma) \mapsto g(y, z, \delta, \gamma)$ is lower semicontinuous.

(POS) positive if $g(y, z, \delta, \gamma) \geq 0$, for all $(y, z, \delta, \gamma) \in \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{d \times d}$.

(CON) convex if $(y, z, \delta, \gamma) \mapsto g(y, z, \delta, \gamma)$ is jointly convex.

(DGC) delta- and gamma-compatible, if there exist $c_1 \in \mathbb{R}$ and $c_2 > 0$ such that, for all $(\delta, \gamma) \in \mathbb{R}^{1 \times d} \times \mathbb{R}^{d \times d}$,

$$g(y, z, \delta, \gamma) \geq c_1 + c_2 (|\delta|^2 + |\gamma|^2)$$

holds for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$.

Remark 2.1.

- (i) Note that (DGC) reflects a penalization of rapid changes in control values. In contrast to Cheridito et al. [14] or Cheridito et al. [15], where the single decomposition parts Δ and Γ were demanded to satisfy certain boundedness, continuity or growth properties, we embed this in (DGC) so that suitable \mathcal{L}^2 -bounds emerge naturally from the problem (2.2).
- (ii) An example of a generator that excludes values of Γ exceeding a certain level by penalization and fits into our setting is given by

$$g(y, z, \delta, \gamma) = \begin{cases} \tilde{g}(y, z, \delta) & \text{if } |\gamma| \leq M \\ +\infty & \text{else} \end{cases},$$

where $M > 0$ and \tilde{g} is any positive, jointly convex and lower semicontinuous generator satisfying $\tilde{g}(y, z, \delta) \geq c_1 + c_2 |\delta|^2$ for constants $c_1 \in \mathbb{R}$ and $c_2 > 0$. This particular choice of g is closely related to the kind of gamma constraints studied in Cheridito et al. [14].

- (iii) Setting the generator $g(\cdot, z, \cdot, \cdot)$ equal to $+\infty$ outside a desired subset of $\mathbb{R}^{1 \times d}$ shows for instance that our framework is flexible enough to comprise shortselling constraints. \blacklozenge

2.2.2. General Properties

The proof of the ensuing Lemma 2.2 can be found in Drapeau et al. [24, Lemma 3.2].

Lemma 2.2. *Let g be a generator satisfying (POS). Assume further that $\mathcal{A}(\xi, g) \neq \emptyset$ and that for the terminal condition ξ holds $\xi^- \in L^1(\mathcal{F}_T)$. Then $\xi \in L^1(\mathcal{F}_T)$ and, for any $(Y, Z) \in \mathcal{A}(\xi, g)$, the control Z is unique and the value process Y is a supermartingale such that $Y_t \geq E[\xi | \mathcal{F}_t]$. Moreover, the unique canonical decomposition of Y is given by*

$$Y = Y_0 + M - A, \quad (2.4)$$

where $M = \int Z dW$ and A is an increasing, predictable, càdlàg process with $A_0 = 0$.

The joint convexity of the generator g immediately yields the following lemma.

Lemma 2.3. *Let g be a generator satisfying (CON). Then, for each $z \in \mathbb{R}^{1 \times d}$, the set $\mathcal{A}(\xi, g, z)$ is convex. Furthermore, from $\mathcal{A}(\xi, g, z^1) \neq \emptyset$ and $\mathcal{A}(\xi, g, z^2) \neq \emptyset$ follows $\mathcal{A}(\xi^\lambda, g, z^\lambda) \neq \emptyset$, for $z^\lambda := \lambda z^1 + (1 - \lambda)z^2$ and $\xi^\lambda := \lambda \xi^1 + (1 - \lambda)\xi^2$ where $\lambda \in [0, 1]$.*

Proof. The first assertion is a direct implication of (CON). As to the latter, it follows from (CON) that $\lambda(Y^1, Z^1) + (1 - \lambda)(Y^2, Z^2) \in \mathcal{A}(\xi^\lambda, g, z^\lambda)$ whenever (Y^1, Z^1) and (Y^2, Z^2) belong to $\mathcal{A}(\xi^1, g, z^1)$ and $\mathcal{A}(\xi^2, g, z^2)$, respectively. \square

For the proof of our main existence theorem we will need an auxiliary result concerning the stability of the set $\Theta(z)$ under convergence in \mathcal{L}^2 , given that the decomposition parts can be uniformly bounded in \mathcal{L}^2 .

Lemma 2.4. *For any $M > 0$ and $z \in \mathbb{R}^{1 \times d}$, the set*

$$\Theta_M(z) = \{Z \in \Theta(z) : \max\{\|\Delta\|_{\mathcal{L}^2}, \| \Gamma \|_{\mathcal{L}^2}\} \leq M\}$$

is closed under convergence in \mathcal{L}^2 . If a sequence $(Z^n) \subset \Theta_M(z)$ with $Z^n = z + \int \Delta^n dt + \int \Gamma^n dW$ converges in \mathcal{L}^2 to some $Z = z + \int \Delta dt + \int \Gamma dW$, then there is a sequence $((\tilde{\Delta}^n, \tilde{\Gamma}^n))$ in the asymptotic convex hull of $((\Delta^n, \Gamma^n))$ converging in $\mathcal{L}^2 \times \mathcal{L}^2$ to (Δ, Γ) .

Proof. First observe that for $Z \in \Theta(z)$ we have

$$|Z_t|^2 \leq 4 \left(|z|^2 + \int_0^t |\Delta_s|^2 ds + \left| \int_0^t \Gamma_s dW_s \right|^2 \right).$$

Hence, for $Z \in \Theta_M(z)$, this in turn yields $E[|Z_t|^2] \leq 4(|z|^2 + \|\Delta\|_{\mathcal{L}^2}^2 + \|\Gamma\|_{\mathcal{L}^2}^2) \leq 4(|z|^2 + 2M^2) := C < \infty$, and hence $\Theta_M(z)$ is a bounded subset of \mathcal{L}^2 , since by Fubini's theorem we obtain that $\|Z\|_{\mathcal{L}^2} \leq \sqrt{TC}$. Consider a sequence $Z^n = z + \int \Delta^n du + \int \Gamma^n dW$ in $\Theta_M(z)$ converging in \mathcal{L}^2 to some process Z . Since $((\Delta^n, \Gamma^n))$ are bounded in $\mathcal{L}^2 \times \mathcal{L}^2$, we can find a sequence $(\tilde{\Delta}^n, \tilde{\Gamma}^n) \in \text{conv}\{(\Delta^n, \Gamma^n), (\Delta^{n+1}, \Gamma^{n+1}), \dots\}$ converging in $\mathcal{L}^2 \times \mathcal{L}^2$ to $(\Delta, \Gamma) \in \mathcal{L}^2 \times \mathcal{L}^2$. Furthermore, it holds that $\|\Delta\|_{\mathcal{L}^2} \vee \|\Gamma\|_{\mathcal{L}^2} \leq M$. Let us

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denote by (\tilde{Z}^n) the respective sequence in the asymptotic convex hull of (Z^n) . From Jensen's inequality we deduce that

$$E \left[\int_0^T \left| \int_0^t (\tilde{\Delta}_s^n - \Delta_s) ds \right|^2 dt \right] \leq TE \left[\int_0^T |\tilde{\Delta}_s^n - \Delta_s|^2 ds \right] \xrightarrow{n \rightarrow \infty} 0,$$

and thus $(\int \tilde{\Delta}^n ds)$ converges to $\int \Delta ds$ in \mathcal{L}^2 . Applying Fubini's theorem and using the Itô isometry yield that

$$E \left[\int_0^T \left| \int_0^t \tilde{\Gamma}_s^n dW_s - \int_0^t \Gamma_s dW_s \right|^2 dt \right] \leq TE \left[\int_0^T |\tilde{\Gamma}_s^n - \Gamma_s|^2 ds \right],$$

where the term on the right-hand side tends to zero by means of the \mathcal{L}^2 -convergence of $(\tilde{\Gamma}^n)$ to Γ . Hence, $(\int \tilde{\Gamma}^n dW)$ converges to $\int \Gamma dW$ in \mathcal{L}^2 . (\tilde{Z}^n) inheriting the \mathcal{L}^2 -convergence to Z from (Z^n) together with the $P \otimes dt$ -uniqueness of \mathcal{L}^2 -limits finally allows us to write the process Z as $Z = z + \int \Delta ds + \int \Gamma dW$, we are done. \square

Lemma 2.4 yields the following compactness result.

Lemma 2.5. *Assume that $\mathcal{A}(\xi, g, z)$ is non-empty for some $z \in \mathbb{R}^{1 \times d}$. Let ξ^- be in $L^1(\mathcal{F}_T)$ and g satisfy (POS), (CON) and (DGC). Then, for any sequence $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g, z)$ of supersolutions satisfying $\sup_n Y_0^n < \infty$, the following holds: There is a sequence (\tilde{Z}^n) in the asymptotic convex hull of (Z^n) that converges in \mathcal{L}^2 to some process $\hat{Z} \in \Theta(z)$. In addition, for all $t \in [0, T]$ it holds $\lim_n \|\tilde{Z}_t^n - \hat{Z}_t\|_{L^2} = 0$.*

Proof. Step 1: Existence of $((\tilde{Y}^n, \tilde{Z}^n))$. \mathcal{L}^2 -convergence of (\tilde{Z}^n) to \hat{Z} . First observe that (2.2) and the supermartingale property of all $\int Z^n dW$ imply that

$$E \left[\int_0^T g_u(Y_u^n, Z_u^n, \Delta_u^n, \Gamma_u^n) du \right] \leq Y_0^n + E[\xi^-] \leq C + E[\xi^-] < \infty, \quad (2.5)$$

where we put $C := \sup_n Y_0^n$. Now, using (2.5) together with (DGC) we estimate

$$\begin{aligned} \|\Delta^n\|_{\mathcal{L}^2}^2 + \|\Gamma^n\|_{\mathcal{L}^2}^2 &= E \left[\int_0^T |\Delta_u^n|^2 du \right] + E \left[\int_0^T |\Gamma_u^n|^2 du \right] \\ &\leq \frac{1}{c_2} E \left[\int_0^T g_u(Y_u^n, Z_u^n, \Delta_u^n, \Gamma_u^n) du \right] - \frac{c_1}{c_2} T \leq \frac{1}{c_2} (C + E[\xi^-] - c_1 T) < \infty. \end{aligned}$$

Since the right-hand above is independent of n , we obtain that $(Z^n) \subset \Theta_M(z)$ with $M := [\frac{1}{c_2}(C + E[\xi^-] - c_1 T)]^{\frac{1}{2}}$ and the arguments within the proof of Lemma 2.4 show that the sequence (Z^n) is uniformly bounded in \mathcal{L}^2 . This in turn guarantees the existence of a sequence (\tilde{Z}^n) in the asymptotic convex hull of (Z^n) that converges to some

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process \hat{Z} in \mathcal{L}^2 and, up to a subsequence, $P \otimes dt$ -almost everywhere.

Step 2: The process \hat{Z} belongs to $\Theta(z)$. The sequence $((\tilde{Y}^n, \tilde{Z}^n))$ lies in $\mathcal{A}(\xi, g, z)$, due to (CON). Moreover, the linearity of the integrals within the Itô decompositions of (Z^n) yields that $\tilde{Z}^n = z + \int \tilde{\Delta}^n du + \int \tilde{\Gamma}^n dW$ where $((\tilde{\Delta}^n, \tilde{\Gamma}^n))$ denotes the corresponding convex combination of the decomposition parts. In addition, $((\tilde{\Delta}^n, \tilde{\Gamma}^n))$ inherits the uniform bound from $((\Delta^n, \Gamma^n))$, that is $\max\{\sup_n \|\tilde{\Delta}^n\|_{\mathcal{L}^2}, \sup_n \|\tilde{\Gamma}^n\|_{\mathcal{L}^2}\} \leq M$. Hence, Lemma 2.4 ensures that \hat{Z} is of the form

$$\hat{Z} = z + \int \hat{\Delta} du + \int \hat{\Gamma} dW,$$

with suitable \mathcal{L}^2 -convergence of the decomposition parts by possibly passing to yet another subsequence in the respective asymptotic convex hulls.

Step 3: $L^2(\mathcal{F}_t)$ -convergence of \tilde{Z}_t^n towards \hat{Z}_t . For all $t \in [0, T]$, the convergence of the respective parts of the decompositions yields that

$$\begin{aligned} \lim_{n \rightarrow \infty} E \left[|\tilde{Z}_t^n - \hat{Z}_t|^2 \right] &\leq 2 \lim_{n \rightarrow \infty} \left(E \left[\int_0^t |\tilde{\Delta}_u^n - \hat{\Delta}_u|^2 du \right] + E \left[\left| \int_0^t (\tilde{\Gamma}_u^n - \hat{\Gamma}_u) dW_u \right|^2 \right] \right) \\ &\leq 2 \lim_{n \rightarrow \infty} \left(\|\tilde{\Delta}^n - \hat{\Delta}\|_{\mathcal{L}^2}^2 + \|\tilde{\Gamma}^n - \hat{\Gamma}\|_{\mathcal{L}^2}^2 \right) = 0, \end{aligned}$$

due to Lemma 2.4. This finishes the proof. \square

2.2.3. Minimality under Constraints

Within the current setup of admissible controls constrained to follow certain dynamics, we study a specific notion of minimality. It corresponds to assessing value processes at fixed times, however given the evolution of the control up to this time, while then deciding which one is preferred among those with identical current state of controls. This differs from the classical notions of global, unconstrained minimality and amounts to a somewhat non-standard optimization problem.

Let us, for each $t \in [0, T]$, define the set of controls attainable at time t by

$$\mathcal{Z}_t := \{Z_t : (Y, Z) \in \mathcal{A}(\xi, g)\}$$

and, for each $V \in \mathcal{Z}_t$ the set of corresponding supersolutions by

$$\mathcal{A}_t(\xi, g, V) := \{(Y, Z) \in \mathcal{A}(\xi, g) : Z_t = V\}.$$

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In the remainder of this work, a major role is played by the operator

$$\mathcal{E}_t^g(\xi, V) := \text{ess inf } \{Y_t : (Y, Z) \in \mathcal{A}_t(\xi, g, V)\} \quad (2.6)$$

which is defined for each $t \in [0, T]$ and $V \in \mathcal{Z}_t$. A supersolution (\hat{Y}, \hat{Z}) is said to be minimal at time $t \in [0, T]$ if

$$\hat{Y}_t = \mathcal{E}_t^g(\xi, \hat{Z}_t).$$

The definition of a supersolution directly yields that $\mathcal{A}_t(\xi^1, g, V) \subseteq \mathcal{A}_t(\xi^2, g, V)$ whenever $\xi^1 \geq \xi^2$. Thus, we immediately obtain monotonicity of the operators $\mathcal{E}_t(\cdot, V)$, that is $\xi^1 \geq \xi^2$ implies $\mathcal{E}_t^g(\xi^1, V) \geq \mathcal{E}_t^g(\xi^2, V)$.

Remark 2.6. The above definition of minimality concerning a priori only the value of control processes at single fixed times may seem inappropriate to the reader at the first glance in terms of its impact on stochastic integrals $\int Z dW$. In this spirit, one might argue that the right notion of minimality at time $t \in [0, T]$ should be the one conditioned on the whole past $Z1_{[0,t]}$ of the control. Corollary 2.10 below states that the two notions in fact coincide. \blacklozenge

For each $t \in [0, T]$, the ensuing Theorem 2.8 provides existence of supersolutions minimal at t , making use of the fact that the set $\{Y_t : (Y, Z) \in \mathcal{A}_t(\xi, g, V)\}$ is directed downwards. Parts of it rely on a version of Helly's theorem which we state here for the sake of completeness. In order to keep this work self-contained, we include the proof given in Heyne [37, Lemma 1.25].

Lemma 2.7. *Let (A^n) be a sequence of increasing positive processes such that the sequence (A_T^n) is bounded in $L^1(\mathcal{F}_T)$. Then, there is a sequence (\tilde{A}^n) in the asymptotic convex hull of (A^n) and an increasing positive integrable process \tilde{A} such that*

$$\lim_{n \rightarrow \infty} \tilde{A}_t^n = \tilde{A}_t, \quad \text{for all } t \in [0, T], \text{ } P\text{-almost surely.}$$

Proof. Let (t_j) be a sequence running through $I := ([0, T] \cap \mathbb{Q}) \cup \{T\}$. Since $(A_{t_1}^n)$ is an L^1 -bounded sequence of positive random variables, due to Delbaen and Schachermayer [18, Lemma A1.1] there exists a sequence $(\tilde{A}^{1,k})$ in the asymptotic convex hull of (A^n) and a random variable \tilde{A}_{t_1} such that $(\tilde{A}_{t_1}^{1,k})$ converges P -almost surely to \tilde{A}_{t_1} . Moreover, Fatou's lemma yields $\tilde{A}_{t_1} \in L^1$. Let $(\tilde{A}^{2,k})$ be a sequence in the asymptotic convex hull of $(\tilde{A}^{1,k})$ such that $(\tilde{A}_{t_2}^{2,k})$ converges P -almost surely to $\tilde{A}_{t_2} \in L^1$ and so on. Then, for $s \in I$, it holds $\tilde{A}_s^{k,k} \rightarrow \tilde{A}_s$ on a set $\hat{\Omega} \subset \Omega$ satisfying $P(\hat{\Omega}) = 1$. The process \tilde{A} is positive, increasing and integrable on I . Thus we may define

$$\hat{A}_t := \lim_{r \downarrow t, r \in I} \tilde{A}_r, \quad t \in [0, T), \quad \hat{A}_T := \tilde{A}_T.$$

We now show that $(\tilde{A}^{k,k})$, henceforth named (\tilde{A}^k) , converges P -almost surely on the continuity points of \hat{A} . To this end, fix $\omega \in \hat{\Omega}$ and a continuity point $t \in [0, T)$ of $\hat{A}(\omega)$. We show that $(\tilde{A}_t^k(\omega))$ is a Cauchy sequence in \mathbb{R} . Fix $\varepsilon > 0$ and set $\delta = \frac{\varepsilon}{11}$. Since t is a continuity point of $\hat{A}(\omega)$, we may choose $p_1, p_2 \in I$ such that $p_1 < t < p_2$

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and $\hat{A}_{p_1}(\omega) - \hat{A}_{p_2}(\omega) < \delta$. By definition of \hat{A} , we may choose $r_1, r_2 \in I$ such that $p_1 < r_1 < t < p_2 < r_2$ and $|\hat{A}_{p_1}(\omega) - \hat{A}_{r_1}(\omega)| < \delta$ and $|\hat{A}_{p_2}(\omega) - \hat{A}_{r_2}(\omega)| < \delta$. Now choose $N \in \mathbb{N}$ such that $|\tilde{A}_{r_1}^m(\omega) - \tilde{A}_{r_1}^n(\omega)| < \delta$, for all $m, n \in \mathbb{N}$ with $m, n \geq N$, and $|\tilde{A}_{r_2}^j(\omega) - \tilde{A}_{r_2}(\omega)| < \delta$ and $|\tilde{A}_{r_1}(\omega) - \tilde{A}_{r_1}^j(\omega)| < \delta$ for $j = m, n$. We estimate

$$|\tilde{A}_t^m(\omega) - \tilde{A}_t^n(\omega)| \leq |\tilde{A}_t^m(\omega) - \tilde{A}_{r_1}^m(\omega)| + |\tilde{A}_{r_1}^m(\omega) - \tilde{A}_{r_1}^n(\omega)| + |\tilde{A}_{r_1}^n(\omega) - \tilde{A}_t^n(\omega)|.$$

For the first and the third term on the right hand side, since A and \bar{A} are increasing, we deduce that $|\tilde{A}_t^m(\omega) - \tilde{A}_{r_1}^m(\omega)| \leq |\tilde{A}_{r_2}^m(\omega) - \tilde{A}_{r_1}^m(\omega)|$ and $|\tilde{A}_t^n(\omega) - \tilde{A}_{r_1}^n(\omega)| \leq |\tilde{A}_{r_2}^n(\omega) - \tilde{A}_{r_1}^n(\omega)|$. Furthermore,

$$\begin{aligned} |\tilde{A}_{r_2}^j(\omega) - \tilde{A}_{r_1}^j(\omega)| &\leq |\tilde{A}_{r_2}^j(\omega) - \tilde{A}_{r_2}(\omega)| + |\tilde{A}_{r_2}(\omega) - \hat{A}_{p_2}(\omega)| \\ &\quad + |\hat{A}_{p_2}(\omega) - \hat{A}_{p_1}(\omega)| + |\hat{A}_{p_1}(\omega) - \tilde{A}_{r_1}(\omega)| + |\tilde{A}_{r_1}(\omega) - \tilde{A}_{r_1}^j(\omega)|, \end{aligned}$$

for $j = m, n$. Combining the previous inequalities yields $|\tilde{A}_t^m(\omega) - \tilde{A}_t^n(\omega)| \leq \varepsilon$, for all $m, n \geq N$. Hence, $(\tilde{A}^k(\omega))$ converges for all continuity points $t \in [0, T)$ of $\hat{A}(\omega)$, for all $\omega \in \hat{\Omega}$. We denote the limit by \tilde{A} .

It remains to be shown that (\tilde{A}^k) also converges for the discontinuity points of \hat{A} . To this end, note that \hat{A} is càdlàg and adapted to our filtration which fulfills the usual conditions. By a well-known result, see for example Karatzas and Shreve [46, Proposition 1.2.26], this implies that the jumps of \hat{A} may be exhausted by a sequence of stopping times (ρ^j) . Applying once more Delbaen and Schachermayer [18, Lemma A1.1] iteratively on the sequences $(\tilde{A}_{\rho^j}^k)_{k \in \mathbb{N}}$, $j = 1, 2, 3, \dots$, and diagonalizing yields the result. \square

Theorem 2.8. *Assume that $\mathcal{A}(\xi, g) \neq \emptyset$ for some $\xi^- \in L^1(\mathcal{F}_T)$ and let g satisfy (LSC), (POS), (CON) and (DGC). Then, for any fixed time $t \in [0, T]$ and random variable $V \in \mathcal{Z}_t$, the set*

$$\left\{ (\hat{Y}, \hat{Z}) \in \mathcal{A}_t(\xi, g, V) : \hat{Y}_t = \mathcal{E}_t^g(\xi, \hat{Z}_t) \right\}$$

is non-empty.

Proof. Step 1: Downward directedness. Since $V \in \mathcal{Z}_t$, there exists a supersolution $(\bar{Y}, \bar{Z}) \in \mathcal{A}_t(\xi, g, V)$ and thus the set in (2.6) is non-empty. In addition, it is directed downwards. Indeed, for two supersolutions $((Y^i, Z^i))_{i=1,2} \subset \mathcal{A}_t(\xi, g, V)$, the pair (Y^*, Z^*) defined by

$$(Y^*, Z^*) = (Y^1, Z^1)1_{[0,t[} + \left(1_{\{Y_t^1 \leq Y_t^2\}}(Y^1, Z^1) + 1_{\{Y_t^1 > Y_t^2\}}(Y^2, Z^2) \right) 1_{[t,T]}$$

is again an element of $\mathcal{A}_t(\xi, g, V)$ with the property $Y_t^* \leq Y_t^1 \wedge Y_t^2$.

Step 2: The candidate control \hat{Z} . Due to the previous step, we can extract a sequence $((\bar{Y}^n, \bar{Z}^n)) \subset \mathcal{A}_t(\xi, g, V)$ such that

$$\lim_{n \rightarrow \infty} \bar{Y}_t^n = \mathcal{E}_t^g(\xi, V) \quad \text{and} \quad \bar{Y}_t^n \leq \bar{Y}_t.$$

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By means of the pair (\bar{Y}, \bar{Z}) we next construct the new sequence $((Y^n, Z^n))$ by

$$\begin{aligned} Y^n &:= \bar{Y}1_{[0,t[} + \bar{Y}^n1_{[t,T]} \\ Z^n &:= \bar{Z}1_{[0,t]} + \bar{Z}^n1_{]t,T]}. \end{aligned}$$

We note that $((Y^n, Z^n)) \subset \mathcal{A}_t(\xi, g, V)$ and $(Z^n) \subset \Theta(\bar{Z}_0)$, since, for all $n \in \mathbb{N}$, it holds $\bar{Y}_t \geq \bar{Y}_t^n$ and $Z_0^n = \bar{Z}_0$, respectively. Because $\sup_n Y_0^n = \bar{Y}_0 < \infty$, Lemma 2.5 assures the existence of a sequence (\tilde{Z}^n) in the asymptotic convex hull of (Z^n) that converges in \mathcal{L}^2 to some admissible process $\hat{Z} \in \Theta(\bar{Z}_0)$, including \mathcal{L}^2 -convergence of the corresponding decomposition parts. In particular, we obtain that

$$\int_0^s \tilde{Z}_u^ndW_u \xrightarrow{n \rightarrow \infty} \int_0^s \hat{Z}_udW_u, \quad \text{for all } s \in [0, T], \text{ } P\text{-almost surely.} \quad (2.7)$$

Moreover, up to a subsequence, $((\tilde{Z}^n, \tilde{\Delta}^n, \tilde{I}^n))$ converges $P \otimes dt$ -almost everywhere towards $(\hat{Z}, \hat{\Delta}, \hat{I})$. Since we also have $L^2(\mathcal{F}_t)$ -convergence of (\tilde{Z}_t^n) to \hat{Z}_t , compare Lemma 2.5, it follows from $\tilde{Z}_t^n = V$ for all $n \in \mathbb{N}$ that

$$\hat{Z}_t = V. \quad (2.8)$$

Step 3: The candidate value process \hat{Y} . If we denote by (\tilde{Y}^n) the sequence in the asymptotic convex hull of (Y^n) corresponding to (\tilde{Z}^n) , then all $(\tilde{Y}^n, \tilde{Z}^n)$ satisfy (2.2) due to (CON). Furthermore, (\tilde{Z}^n) is uniformly bounded in \mathcal{L}^2 . Let \tilde{A}^n denote the increasing, predictable process of finite variation stemming from the decomposition of $\tilde{Y}^n = \bar{Y}_0 + \tilde{M}^n - \tilde{A}^n$ given in Lemma 2.2. Since all $\int \tilde{Z}^ndW$ are true martingales and g satisfies (POS), the decomposition (2.4) yields

$$E[\tilde{A}_T^n] \leq \bar{Y}_0 + E[\xi^-] < \infty,$$

as we assumed ξ^- to be an element of $L^1(\mathcal{F}_T)$. Now a version of Helly's theorem, see Lemma 2.7, yields the existence of a sequence in the asymptotic convex hull of (\tilde{A}^n) , again denoted by the previous expression, and of an increasing positive integrable process \tilde{A} such that $\lim_{n \rightarrow \infty} \tilde{A}_s^n = \tilde{A}_s$, for all $s \in [0, T]$, P -almost surely. We pass to the corresponding sequence on the side of (\tilde{Y}^n) and (\tilde{Z}^n) , define the process \tilde{Y} pointwise for all $s \in [0, T]$ by $\tilde{Y}_s := \lim_{n \rightarrow \infty} \tilde{Y}_s^n = \bar{Y}_0 + \int_0^s \hat{Z}_udW_u - \tilde{A}_s$, and observe that it fulfills $\tilde{Y}_t = \mathcal{E}_t^g(\xi, V)$ by construction. However, since \tilde{Y} is not necessarily càdlàg, we define our candidate value process \hat{Y} by $\hat{Y}_s := \lim_{r \downarrow s, r \in \mathbb{Q}} \tilde{Y}_r$, for all $s \in [0, T)$ and $\hat{Y}_T := \xi$. The continuity of $\int \hat{Z}dW$ yields that

$$\hat{Y}_s = \bar{Y}_0 + \int_0^s \hat{Z}_udW_u - \lim_{r \downarrow s, r \in \mathbb{Q}} \tilde{A}_r. \quad (2.9)$$

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Since jump times of càdlàg processes² can be exhausted by a sequence of stopping times $(\sigma_j) \subset \mathcal{T}$, compare Karatzas and Shreve [46, Proposition 1.2.26], which coincide with the jump times of \tilde{A} , we conclude that

$$\hat{Y} = \tilde{Y}, \quad P \otimes dt\text{-almost everywhere.} \quad (2.10)$$

Furthermore, \tilde{A} increasing implies that $\hat{A}_s := \lim_{r \downarrow s, r \in \mathbb{Q}} \tilde{A}_r \geq \tilde{A}_s$, for all $s \in [0, T]$ which, together with (2.9), in turn yields that

$$\hat{Y}_s \leq \tilde{Y}_s, \quad \text{for all } s \in [0, T]. \quad (2.11)$$

Given that (\hat{Y}, \hat{Z}) satisfies (2.2), we could by means of (2.8) conclude that $(\hat{Y}, \hat{Z}) \in \mathcal{A}_t(\xi, g, V)$ and thus $\hat{Y}_t \geq \mathcal{E}_t^g(\xi, V) = \tilde{Y}_t$ which, combined with (2.11), would imply $\hat{Y}_t = \mathcal{E}_t^g(\xi, V) = \mathcal{E}_t^g(\xi, \hat{Z}_t)$ and thereby finish the proof.

Step 4: Verification. As to the remaining verification, we deduce from (2.10) the existence of a set $A \in \mathcal{F}_T$, $P(A) = 1$ with the following property. For all $\omega \in A$, there exists a Lebesgue measurable set $\mathcal{I}(\omega) \subset [0, T]$ of measure T such that $\tilde{Y}_s^n(\omega) \rightarrow \hat{Y}_s(\omega)$, for all $s \in \mathcal{I}(\omega)$. We suppress the dependence of \mathcal{I} on ω and recall however that in the following r and s may depend on ω . For $r, s \in \mathcal{I}$ with $r \leq s$ holds

$$\begin{aligned} \hat{Y}_r - \int_r^s g_u(\hat{Y}_u, \hat{Z}_u, \hat{\Delta}_u, \hat{F}_u) du + \int_r^s \hat{Z}_u dW_u \\ \geq \limsup_n \left(\tilde{Y}_r^n - \int_r^s g_u(\tilde{Y}_u^n, \tilde{Z}_u^n, \tilde{\Delta}_u^n, \tilde{F}_u^n) du + \int_r^s \tilde{Z}_u^n dW_u \right) \end{aligned} \quad (2.12)$$

by means of (2.7), the $P \otimes dt$ -almost-everywhere convergence of $((\tilde{Y}^n, \tilde{Z}^n, \tilde{\Delta}^n, \tilde{F}^n))$ to $(\hat{Y}, \hat{Z}, \hat{\Delta}, \hat{F})$, the property (LSC) and Fatou's lemma. Using $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{A}(\xi, g)$, for all $n \in \mathbb{N}$, (2.12) can be further estimated by

$$\hat{Y}_r - \int_r^s g_u(\hat{Y}_u, \hat{Z}_u, \hat{\Delta}_u, \hat{F}_u) du + \int_r^s \hat{Z}_u dW_u \geq \limsup_n \tilde{Y}_s^n = \hat{Y}_s. \quad (2.13)$$

Whenever $r, s \in \mathcal{I}^c$ with $r \leq s$, we approximate both times from the right by sequences $(r^n) \subset \mathcal{I}$ and $(s^n) \subset \mathcal{I}$, respectively, such that $r^n \leq s^n$. Since (2.13) holds for all r^n and s^n , the claim follows from the right-continuity of \hat{Y} and the continuity of all appearing integrals, which finally concludes the proof. \square

²Note that as an increasing process, \tilde{A} is in particular a submartingale and thus its right- and left-hand limits exist, compare Karatzas and Shreve [46, Proposition 1.3.14]. Consequently, the process $\lim_{s \downarrow \cdot, s \in \mathbb{Q}} \tilde{A}_s$ is càdlàg.

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For each $t \in [0, T]$ and $V \in \mathcal{Z}_t$, the set

$$T(t, V) := \left\{ (\hat{Y}, \hat{Z}) \in \mathcal{A}_t(\xi, g, V) : \hat{Y}_t = \mathcal{E}_t^g(\xi, \hat{Z}_t) \right\} \quad (2.14)$$

collects the corresponding minimal supersolutions from Theorem 2.8. As our main assertion concerns only existence of minimal elements and not uniqueness, it is sufficient for our purpose to work with representatives, denoted by $(T^y(t, V), T^z(t, V)) \in \mathcal{A}_t(\xi, g, V)$, for the remainder of this work. Observe that each of the aforementioned naturally satisfies $T_t^y(t, V) = \mathcal{E}_t^g(\xi, V)$. Since an essential infimum is P -almost surely unique by definition, the mapping $\xi \mapsto \mathcal{E}_t^g(\xi, V)$ is well-defined for each $t \in [0, T]$ and $V \in \mathcal{Z}_t$. A straightforward inductive adaptation of the preceding theorem allows us to formulate the following generalization.

Theorem 2.9. *Under the assumptions of Theorem 2.8, the following holds. For each $N \in \mathbb{N}$ and $\{t_1, \dots, t_N\} \subset [0, T]$ with $t_1 < t_2 < \dots < t_N$, the set $\{(\hat{Y}, \hat{Z}) \in \mathcal{A}(\xi, g) : \hat{Y}_{t_i} = \mathcal{E}_{t_i}^g(\xi, \hat{Z}_{t_i}), i = 1, \dots, N\}$ is non-empty, that is, for any finite subset of $[0, T]$ there exists a supersolution minimal at exactly these times.*

Proof. Recall that $\mathcal{A}(\xi, g) \neq \emptyset$ and thus there exists $(Y, Z) \in \mathcal{A}(\xi, g)$. Since by Theorem 2.8 the set $T(t_1, Z_{t_1})$ defined in (2.14) is non-empty, the process

$$\begin{aligned} Y^1 &:= Y1_{[0, t_1[} + T^y(t_1, Z_{t_1})1_{[t_1, T]} \\ Z^1 &:= Z1_{[0, t_1]} + T^z(t_1, Z_{t_1})1_{[t_1, T]} \end{aligned}$$

belongs to $\mathcal{A}(\xi, g)$ and is minimal at time t_1 . Again, the set $T(t_2, Z_{t_2}^1)$ being non-empty by means of Theorem 2.8 allows us to define (Y^2, Z^2) analogously making use of (Y^1, Z^1) . More precisely, we set $Y^2 := Y^11_{[0, t_2[} + T^y(t_2, Z_{t_2}^1)1_{[t_2, T]}$ and $Z^2 := Z^11_{[0, t_2]} + T^z(t_2, Z_{t_2}^1)1_{[t_2, T]}$, and observe that (Y^2, Z^2) is an element of $\mathcal{A}(\xi, g)$ that is minimal at times t_1 and t_2 . Iterating this procedure and setting $t_{N+1} := T$, the pair

$$\begin{aligned} \hat{Y} &:= Y1_{[0, t_1[} + \sum_{i=1}^N T^y(t_i, \hat{Z}_{t_i})1_{[t_i, t_{i+1}[}, \quad \hat{Y}_T = \xi, \\ \hat{Z} &:= Z1_{[0, t_1]} + \sum_{i=1}^N T^z(t_i, \hat{Z}_{t_i})1_{[t_i, t_{i+1}]} \end{aligned}$$

is a well-defined element of $\mathcal{A}(\xi, g)$ and satisfies the desired minimality criterion at all $\{t_i\}_{i=1, \dots, N}$. \square

Notice that the proof of Theorem 2.8 gives insight into the structure of minimizing sequences so that the ensuing corollary is formulated in the spirit of Remark 2.6.

Corollary 2.10. *For all $t \in [0, T]$, all $V \in \mathcal{Z}_t$ and all $(\bar{Y}, \bar{Z}) \in \mathcal{A}_t(\xi, g, V)$, it holds $\mathcal{E}_t^g(\xi, V) = \text{ess inf}\{Y_t : (Y, Z) \in \mathcal{A}_t(\xi, g, V) \text{ and } Z1_{[0, t]} = \bar{Z}1_{[0, t]}\}$.*

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Proof. Obviously, (\leq) is always satisfied in the assertion. Note on the other hand that, for any $(Y, Z) \in \mathcal{A}_t(\xi, g, V)$, we can construct $(Y^*, Z^*) \in \mathcal{A}_t(\xi, g, V)$ satisfying $Z^*1_{[0,t]} = \bar{Z}1_{[0,t]}$ and $Y_t^* \leq Y_t$. Indeed, we simply put

$$(Y^*, Z^*) = (\bar{Y}, \bar{Z})1_{[0,t]} + \left(1_{\{\bar{Y}_t \leq Y_t\}}(\bar{Y}, \bar{Z}) + 1_{\{\bar{Y}_t > Y_t\}}(Y, Z)\right)1_{[t,T]}.$$

Thus, any sequence $((Y^n, Z^n)) \subset \mathcal{A}_t(\xi, g, V)$ satisfying $Y_t^n \downarrow \mathcal{E}_t^g(\xi, V)$ may be replaced by $((\bar{Y}^n, \bar{Z}^n))$ with $\bar{Z}^n1_{[0,t]} = \bar{Z}1_{[0,t]}$ and the convergence still holds. Consequently, the essential infimum of elements of the latter type can be only smaller than $\mathcal{E}_t^g(\xi, V)$. \square

Convexity of the mapping $(\xi, z) \mapsto \mathcal{E}_0^g(\xi, z)$ is provided by the following lemma.

Lemma 2.11. *Under the assumptions of Theorem 2.8, the operator $\mathcal{E}_0^g(\cdot, \cdot)$ is jointly convex.*

Proof. For $z^1, z^2 \in \mathbb{R}^{1 \times d}$ and $\xi^1, \xi^2 \in L^0(\mathcal{F}_T)$, the negative parts of which are integrable, assume that $\mathcal{A}(\xi^1, g, z^1) \neq \emptyset$ and $\mathcal{A}(\xi^2, g, z^2) \neq \emptyset$, as otherwise convexity trivially holds. For $\lambda \in [0, 1]$ we set $z^\lambda := \lambda z^1 + (1 - \lambda)z^2$ and $\xi^\lambda := \lambda \xi^1 + (1 - \lambda)\xi^2$ so that Lemma 2.3 implies $\mathcal{A}(\xi^\lambda, g, z^\lambda) \neq \emptyset$. By Theorem 2.8, there exist (Y^1, Z^1) and (Y^2, Z^2) in $\mathcal{A}(\xi^1, g, z^1)$ and $\mathcal{A}(\xi^2, g, z^2)$, respectively, such that $Y_0^1 = \mathcal{E}_0^g(\xi^1, z^1)$ and $Y_0^2 = \mathcal{E}_0^g(\xi^2, z^2)$. Since $(\bar{Y}, \bar{Z}) := \lambda(Y^1, Z^1) + (1 - \lambda)(Y^2, Z^2)$ is an element of $\mathcal{A}(\xi^\lambda, g, z^\lambda)$ due to (CON), it holds $\mathcal{E}_0^g(\xi^\lambda, z^\lambda) \leq \bar{Y}_0$ by definition of the operator \mathcal{E}_0^g . \square

2.2.4. Stability Results

Next, we show that the non-linear operator $\xi \mapsto \mathcal{E}_0^g(\xi, z)$ exhibits stability properties such as monotone convergence or the Fatou property, which, under slightly stronger assumptions, may also be extended to arbitrary times. Furthermore, we prove L^1 -lower semicontinuity of $\mathcal{E}_0^g(\cdot, z)$, a property that is crucial for the duality in Section 2.3. In addition, for generators that are independent of y , we introduce an operator $\tilde{\mathcal{E}}_0^g(\cdot, z)$ related to an a priori weaker notion of minimality at time zero and subsequently show that it coincides with $\mathcal{E}_0^g(\cdot, z)$, thereby gaining more insight as to how the problem might be formulated in terms of approximating sequences.

The following theorem establishes monotone convergence and the Fatou property of $\mathcal{E}_0^g(\cdot, z)$. Similar results in the unconstrained case have been obtained in Drapeau et al. [24, Theorem 4.7].

Theorem 2.12. *For $z \in \mathbb{R}^{1 \times d}$ and g a generator fulfilling (LSC), (POS), (CON) and (DGC), and (ξ_n) a sequence in $L^0(\mathcal{F}_T)$ such that $(\xi_n^-) \subset L^1(\mathcal{F}_T)$, the following holds.*

- *Monotone convergence:* If (ξ_n) is increasing P -almost surely to $\xi \in L^0(\mathcal{F}_T)$, then it holds $\lim_{n \rightarrow \infty} \mathcal{E}_0^g(\xi_n, z) = \mathcal{E}_0^g(\xi, z)$.
- *Fatou's lemma:* If $\xi_n \geq \eta$, for all $n \in \mathbb{N}$, where $\eta \in L^1(\mathcal{F}_T)$, then it holds $\mathcal{E}_0^g(\liminf_n \xi_n, z) \leq \liminf_n \mathcal{E}_0^g(\xi_n, z)$.

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Proof. Monotone convergence: First, by monotonicity the limit $\bar{Y}_0 := \lim_n \mathcal{E}_0^g(\xi_n, z)$ exists and satisfies $\bar{Y}_0 \leq \mathcal{E}_0^g(\xi, z)$. Other than in the trivial case of $+\infty = \bar{Y}_0 \leq \mathcal{E}_0^g(\xi, z)$ we have $\mathcal{A}(\xi_n, g, z) \neq \emptyset$, for all $n \in \mathbb{N}$. Furthermore, since $(\xi_n^-) \subset L^1(\mathcal{F}_T)$, Theorem 2.8 yields the existence of supersolutions $(Y^n, Z^n) \in \mathcal{A}(\xi_n, g, z)$ fulfilling $Y_0^n = \mathcal{E}_0^g(\xi_n, z)$, for all $n \in \mathbb{N}$. In particular, we have that $Y_0^n \leq \bar{Y}_0$ and $\xi_n^- \leq \xi_1^-$, for all $n \in \mathbb{N}$, and thus $\sup_n Y_0^n < \infty$ as well as $\sup_n E[\xi_n^-] < \infty$. Hence, arguments analogous to the ones used in Lemma 2.5 and the proof of Theorem 2.8 directly translate to the present setting and provide both a candidate control $\hat{Z} \in \Theta(z)$ to which (\hat{Z}^n) converges and a corresponding $\hat{Y}_t := \lim_n \hat{Y}_t^n$, and ensure that (\hat{Y}, \hat{Z}) belongs to $\mathcal{A}(\xi, g, z)$, where $\hat{Y} := \lim_{s \in \mathbb{Q}, s \downarrow} \hat{Y}_s$ on $[0, T)$ and $\hat{Y}_T := \xi$. In particular, we obtain $\hat{Y}_0 \leq \bar{Y}_0 = \bar{Y}_0$. Thus, as $\mathcal{A}(\xi, g, z) \neq \emptyset$ and $\xi^- \in L^1(\mathcal{F}_T)$, there exists $(Y, Z) \in \mathcal{A}(\xi, g, z)$ such that $Y_0 = \mathcal{E}_0^g(\xi, z)$. By minimality of (Y, Z) , however, this entails $Y_0 \leq \hat{Y}_0 \leq \bar{Y}_0$ and we conclude that $\lim_{n \rightarrow \infty} \mathcal{E}_0^g(\xi_n, z) = \mathcal{E}_0^g(\xi, z)$.

Fatou's lemma: If we define $\zeta_n := \inf_{k \geq n} \xi_k$, then $\xi_k \geq \eta$ for all $k \in \mathbb{N}$ implies $\zeta_n \geq \eta$ for all $n \in \mathbb{N}$ which in turn gives $(\zeta_n^-) \subset L^1(\mathcal{F}_T)$, and thus the monotone convergence established above can be used exactly as in Drapeau et al. [24, Theorem 4.7] to obtain the assertion. \square

As a consequence of the monotone convergence property we obtain the ensuing theorem providing L^1 -lower semicontinuity of the operator $\mathcal{E}_0^g(\cdot, z)$. The proof goes along the lines of Drapeau et al. [24, Theorem 4.9] and is thus omitted here.

Theorem 2.13. *Let $z \in \mathbb{R}^{1 \times d}$ and g be a generator fulfilling (LSC), (POS), (CON) and (DGC). Then $\mathcal{E}_0^g(\cdot, z)$ is L^1 -lower semicontinuous.*

Under the additional assumption that $\mathcal{A}_t(\xi, g, V) \neq \emptyset$ for a sequence (ξ_n) increasing to ξ , monotone convergence translates to arbitrary times $t \in [0, T]$, that is to the operator $\mathcal{E}_t^g(\cdot, V)$, as illustrated by the following theorem. Likewise, given that $\mathcal{A}_t(\liminf_n \xi_n, g, V) \neq \emptyset$, also the Fatou property holds.

Theorem 2.14. *Let a terminal condition $\xi \in L^0(\mathcal{F}_T)$, an \mathcal{F}_t -measurable random variable V such that $\mathcal{A}_t(\xi, g, V) \neq \emptyset$, and a sequence (ξ_n) in $L^0(\mathcal{F}_T)$ such that $(\xi_n^-) \subset L^1(\mathcal{F}_T)$ be given. If the generator g satisfies (LSC), (POS), (CON) and (DGC), the following holds.*

- *Monotone convergence:* If (ξ_n) is increasing P -almost surely to $\xi \in L^0(\mathcal{F}_T)$, then it holds $\lim_{n \rightarrow \infty} \mathcal{E}_t^g(\xi_n, V) = \mathcal{E}_t^g(\xi, V)$, P -almost surely.
- *Fatou's lemma:* Suppose $\mathcal{A}_t(\liminf_n \xi_n, g, V) \neq \emptyset$. If $\xi_n \geq \eta$, for all $n \in \mathbb{N}$, where $\eta \in L^1(\mathcal{F}_T)$, then it holds $\mathcal{E}_t^g(\liminf_n \xi_n, V) \leq \liminf_n \mathcal{E}_t^g(\xi_n, V)$, P -almost surely.

Proof. Monotone convergence: By monotonicity the limit $\bar{Y}_t := \lim_n \mathcal{E}_t^g(\xi_n, V)$ exists and satisfies $\bar{Y}_t \leq \mathcal{E}_t^g(\xi, V)$. Moreover, $\mathcal{A}_t(\xi, g, V) \neq \emptyset$ in combination with (ξ_n) being increasing to ξ implies $\mathcal{A}(\xi_n, g, z) \neq \emptyset$, for all $n \in \mathbb{N}$. Since $(\xi_n^-) \subset L^1(\mathcal{F}_T)$, Theorem 2.8 yields the existence of supersolutions $(\bar{Y}^n, \bar{Z}^n) \in \mathcal{A}_t(\xi_n, g, V)$ fulfilling $\bar{Y}_t^n = \mathcal{E}_t^g(\xi_n, V)$, for all $n \in \mathbb{N}$, and, as $\xi^- \leq \xi_1^- \in L^1(\mathcal{F}_T)$, also of $(Y, Z) \in \mathcal{A}_t(\xi, g, V)$ such that

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$Y_t = \mathcal{E}_t^g(\xi, V)$. Analogously to the proof of Theorem 2.12, the crucial part consists in finding a suitable sequence $((Y^n, Z^n))$ satisfying $\sup_n Y_0^n < \infty$ and $\sup_n E[\xi_n^-] < \infty$, as well as $Y_t^n = \mathcal{E}_t^g(\xi_n, V)$. Making use of $Y_t = \mathcal{E}_t^g(\xi, V) \geq \bar{Y}_t^n$, the pairs

$$Y^n := Y1_{[0,t]} + \bar{Y}^n 1_{[t,T]} \quad , \quad Z^n := Z1_{[0,t]} + \bar{Z}^n 1_{[t,T]}$$

exhibit the required properties. In particular, $\lim_n Y_t^n = \bar{Y}_t$. Hence, as in the proof at time zero, we obtain $(\hat{Y}, \hat{Z}) \in \mathcal{A}_t(\xi, g, V)$ such that $\hat{Y}_t \leq \bar{Y}_t$. By minimality at time t of (Y, Z) , however, we obtain $Y_t \leq \hat{Y}_t \leq \bar{Y}_t$ and conclude that $\lim_{n \rightarrow \infty} \mathcal{E}_t^g(\xi_n, V) = \mathcal{E}_t^g(\xi, V)$, P -almost surely.

Fatou's lemma: Since we assumed $\mathcal{A}_t(\liminf_n \xi_n, g, V)$ to be non-empty, the monotone convergence derived above can be used to prove the assertion exactly as in the proof of Theorem 2.12. \square

Assumption 2.15. *For the remainder of Chapter 2 we assume that generators are independent of y , that is $g_u(y, z, \delta, \gamma) = g_u(z, \delta, \gamma)$.*

We continue by discussing a weak formulation of minimality at time zero in terms of sequences approximating the terminal condition ξ from below in $L^1(\mathcal{F}_T)$. Recall that each $z \in \mathbb{R}^{1 \times d}$ induces its set $\Theta(z)$ of admissible controls with initial value z . For ξ a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$, consider the operator $\tilde{\mathcal{E}}_0^g(\xi, z)$ defined by

$$\tilde{\mathcal{E}}_0^g(\xi, z) := \inf \left\{ y \in \mathbb{R} : \text{there exists } (Z^n) \subset \Theta(z) \text{ such that} \right. \\ \left. \left\| \left(y - \int_0^T g_u(Z_u^n, \Delta_u^n, \Gamma_u^n) du + \int_0^T Z_u^n dW_u - \xi \right)^- \right\|_{L^1} \xrightarrow{n \rightarrow \infty} 0 \right\}. \quad (2.15)$$

The next proposition shows that the infimum within the above expression is attained in \mathbb{R} and that the weak and strong formulation of minimality coincide.

Proposition 2.16. *Let g be a generator satisfying (LSC), (POS), (CON) and (DGC). If for ξ satisfying $\xi^- \in L^1(\mathcal{F}_T)$ the set $\mathcal{A}(\xi, g, z)$ is non-empty, then $\tilde{\mathcal{E}}_0^g(\xi, z) \in \mathbb{R}$, the infimum in (2.15) is attained and we have the equality*

$$\tilde{\mathcal{E}}_0^g(\xi, z) = \mathcal{E}_0^g(\xi, z).$$

Proof. First observe that by means of (2.2) any supersolution $(Y, Z) \in \mathcal{A}(\xi, g, z)$ satisfies $Y_0 - \int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^T Z_u dW_u \geq \xi$ and hence

$$\left(Y_0 - \int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^T Z_u dW_u - \xi \right)^- = 0.$$

Thus, the set over which the infimum in (2.15) is taken is non-empty and $\tilde{\mathcal{E}}_0^g(\xi, z) \leq \mathcal{E}_0^g(\xi, z)$ holds true. Next, for $y \in \mathbb{R}$ belonging to this set we fix $\varepsilon > 0$ and choose n

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sufficiently large so as to bound the appearing L^1 -norm by ε . Using the positivity of all $\int g(Z^n, \Delta^n, \Gamma^n) du$ we may further estimate the negative part by

$$\begin{aligned} \varepsilon &\geq E \left[\left(y + \int_0^T Z^n dW - \xi \right)^- \right] \\ &= E \left[-y - \int_0^T Z^n dW + \xi \right] + E \left[\left(y + \int_0^T Z^n dW - \xi \right)^+ \right]. \end{aligned}$$

The positivity of the second term on the right and all $\int Z^n dW$ being martingales then lead to $y \geq -E[\xi^-] - \varepsilon$. Letting ε tend to zero, we derive the boundedness from below of $\tilde{\mathcal{E}}_0(\xi, z)$ as the infimum over all such y . Let now (y_m) be a decreasing sequence converging to $\tilde{\mathcal{E}}_0^g(\xi, z)$. We may choose a subsequence, again denoted by (y_m) , satisfying $|y_m - \tilde{\mathcal{E}}_0^g(\xi, z)| \leq 1/m$. For $m = 1$, we extract out of the corresponding sequence $(Z^{1,n})_n$ a control $\bar{Z}^1 := Z^{1,n_1}$ fulfilling $\|(y_1 - \int_0^T g_u(\bar{Z}_u^1, \bar{\Delta}_u^1, \bar{\Gamma}_u^1) du + \int_0^T \bar{Z}_u^1 dW_u - \xi)^-\|_{L^1} \leq 1$, for $m = 2$ and correspondingly chosen $\bar{Z}^2 := Z^{2,n_2}$ the last expression has to be bounded by $1/2$ and so on, that is in the m -th step we set the bound to $1/m$. Following this procedure yields a weak superreplicating sequence $(\bar{Z}^m)_m$ for $\tilde{\mathcal{E}}_0^g(\xi, z)$ as a simple consequence of the triangular inequality by which we deduce that $\tilde{\mathcal{E}}_0^g(\xi, z)$ is attained.

It remains to show the reverse inequality $\mathcal{E}_0^g(\xi, z) \leq \tilde{\mathcal{E}}_0^g(\xi, z)$. Since the infimum is attained, we may use the controls (Z^n) appearing in the weak formulation in order to define the sequence of terminal conditions $(\tilde{\xi}_n)$ by

$$\tilde{\xi}_n := \tilde{\mathcal{E}}_0^g(\xi, z) - \int_0^T g_u(Z_u^n, \Delta_u^n, \Gamma_u^n) du + \int_0^T Z_u^n dW_u. \quad (2.16)$$

Setting $\xi_n := \tilde{\xi}_n \wedge \xi$ we observe that (2.15) entails L^1 -convergence of (ξ_n) to ξ . Therefore the L^1 -lower semicontinuity of $\mathcal{E}_0^g(\cdot, z)$, compare Theorem 2.13, together with $\xi_n \leq \tilde{\xi}_n$ and the monotonicity of $\mathcal{E}_0^g(\cdot, z)$ yields

$$\mathcal{E}_0^g(\xi, z) = \mathcal{E}_0^g(\liminf_n \xi_n, z) \leq \liminf_n \mathcal{E}_0^g(\xi_n, z) \leq \liminf_n \mathcal{E}_0^g(\tilde{\xi}_n, z) \leq \tilde{\mathcal{E}}_0^g(\xi, z), \quad (2.17)$$

where the last inequality follows from $\tilde{\mathcal{E}}_0^g(\xi, z)$ being by (2.16) an initial value of solutions of BSDEs with terminal conditions $\tilde{\xi}_n$. \square

We complete this section concerning stability results by deriving joint lower semicontinuity of $\mathcal{E}_0^g(\cdot, \cdot)$, given that g is uniformly Lipschitz continuous in z . The proof of this statement strongly relies on the equivalence of the weak and strong formulation derived in Proposition 2.16, since joint lower semicontinuity of $\mathcal{E}_0^g(\cdot, \cdot)$ is thereby of course equivalent to that of $\tilde{\mathcal{E}}_0^g(\cdot, \cdot)$.

Lemma 2.17. *Let g be a generator satisfying (LSC), (POS), (CON) and (DGC). If g is in addition Lipschitz continuous in z uniformly on $\Omega \times [0, T] \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{d \times d}$, then the*

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function $(\xi, z) \mapsto \tilde{\mathcal{E}}_0^g(\xi, z)$ is lower semicontinuous³ on its domain, that is on $\{(\xi, z) \in L^0(\mathcal{F}_T) \times \mathbb{R}^{1 \times d} : \xi^- \in L^1(\mathcal{F}_T) \text{ and } \mathcal{A}(\xi, g, z) \neq \emptyset\}$.

Proof. By Proposition 2.16, it holds $\tilde{\mathcal{E}}_0^g(\xi, z) \in \mathbb{R}$. For any sequence $((\tilde{\xi}_n, \tilde{z}_n))$ within the domain that converges to (ξ, z) , let us consider the subsequence $((\xi_n, z_n))$ of $((\tilde{\xi}_n, \tilde{z}_n))$ such that $\liminf_n \tilde{\mathcal{E}}_0^g(\tilde{\xi}_n, \tilde{z}_n) = \lim_n \tilde{\mathcal{E}}_0^g(\xi_n, z_n) =: y$ and assume in contrast to the assertion of lower semicontinuity that

$$\tilde{\mathcal{E}}_0^g(\xi, z) > y. \quad (2.18)$$

The elements in consideration (ξ_n, z_n) belonging to the domain of $\mathcal{E}_0^g(\cdot, \cdot)$ together with Proposition 2.16 and Theorem 2.8 implies the existence of a sequence $((Y^n, Z^n))$ such that $(Y^n, Z^n) \in \mathcal{A}(\xi_n, g, z_n)$ and $Y_0^n = \mathcal{E}_0^g(\xi_n, z_n) = \tilde{\mathcal{E}}_0^g(\xi_n, z_n)$. In particular, by means of (2.2) it holds

$$\left(\tilde{\mathcal{E}}_0^g(\xi_n, z_n) - \int_0^T g_u(Z_u^n, \Delta_u^n, \Gamma_u^n) du + \int_0^T Z_u^n dW - \xi_n \right)^- = 0. \quad (2.19)$$

We prove that it is possible to get arbitrarily close from below to ξ in the L^1 -sense, starting in y and using a sequence $(\bar{Z}^n) \subset \Theta(z)$. This would imply $\tilde{\mathcal{E}}_0^g(\xi, z) \leq y$, a contradiction to (2.18), and finish the proof. Let to this end an arbitrary $\varepsilon > 0$ be given. We set $\bar{Z}^n := z + \int_0^T \Delta^n du + \int_0^T \Gamma^n dW$ and note that $(\bar{Z}^n) \subset \Theta(z)$ as well as $\bar{Z}^n - Z^n = z - z_n$ due to $Z_0^n = z_n$. We choose $n = n(\varepsilon)$ large enough to ensure that

$$\|\xi - \xi_n\|_{L^1} \vee |\tilde{\mathcal{E}}_0^g(\xi_n, z_n) - y| \vee |z - z_n| \leq \frac{\varepsilon}{4(1 + (\sqrt{dT} \vee TL))}, \quad (2.20)$$

where L is the uniform Lipschitz constant of g , that is $|g(z, \cdot, \cdot) - g(z', \cdot, \cdot)| \leq L|z - z'|$. Note that by construction the decomposition parts of (\bar{Z}^n) and (Z^n) are equal, that is $(\bar{\Delta}^n, \bar{\Gamma}^n) = (\Delta^n, \Gamma^n)$. Making use of the estimate

$$\int_0^T g_u(\bar{Z}_u^n, \Delta_u^n, \Gamma_u^n) du \leq \int_0^T g_u(Z_u^n, \Delta_u^n, \Gamma_u^n) du + TL|z_n - z|$$

³Here, lower semicontinuity of $(\xi, z) \mapsto \tilde{\mathcal{E}}_0^g(\xi, z)$ is to be understood with respect to the L^1 -norm in the first argument.

in addition with the decomposition $\int_0^T \bar{Z}_u^n dW_u = \int_0^T Z_u^n dW_u + (z - z_n)W_T$, we obtain

$$\begin{aligned} & \left(y - \int_0^T g_u(\bar{Z}_u^n, \bar{\Delta}_u^n, \bar{\Gamma}_u^n) du + \int_0^T \bar{Z}_u^n dW_u - \xi \right)^- \\ & \leq \left(\tilde{\mathcal{E}}_0^g(\xi_n, z_n) - \int_0^T g_u(Z_u^n, \Delta_u^n, \Gamma_u^n) du + \int_0^T Z_u^n dW_u - \xi_n \right)^- \\ & \quad + |\tilde{\mathcal{E}}_0^g(\xi_n, z_n) - y| + |\xi - \xi_n| + TL|z_n - z| + |z - z_n||W_T|. \end{aligned}$$

Taking expectation on both sides together with (2.19) and (2.20) finally yields that the left-hand side of the above is bounded in $L^1(\mathcal{F}_T)$ by ε . \square

The proof above illustrates that the uniform Lipschitz continuity of g in z may also be replaced by uniform Hölder continuity and the statement is still valid. In addition, whenever terminal conditions are of the form $\xi = \varphi(W_T)$ for a Lipschitz continuous function $\varphi : \mathbb{R}^{d \times 1} \rightarrow \mathbb{R}$, the preceding lemma in particular implies that $(x, z) \mapsto \tilde{\mathcal{E}}_0^g(\xi^x, z) := \tilde{\mathcal{E}}_0^g(\varphi(x + W_T), z)$ is lower semicontinuous and thus, by Proposition 2.16, so is $(x, z) \mapsto \mathcal{E}_0^g(\xi^x, z)$. Indeed, $\|\varphi(x_n + W_T) - \varphi(x + W_T)\|_{L^1} \leq L|x_n - x|$ shows that (ξ^{x_n}) converges in L^1 to ξ^x whenever (x_n) converges to x .

2.3. Duality under Constraints

The objective of this section is to construct solutions of constrained BSDEs via duality and, for the case of a quadratic generator, to obtain an explicit form for \mathcal{E}_0^* , the Fenchel-Legendre transform of \mathcal{E}_0^g . Recall that our generator g is supposed to be independent of y throughout this section and we assume that it satisfies (LSC), (POS), (CON) and (DGC). Let us further fix some $z \in \mathbb{R}^{1 \times d}$ as initial value of the controls and set $\mathcal{E}_0^g(\cdot) := \mathcal{E}_0^g(\cdot, z)$ for the remainder of this section. In order to simplify the notation we will just write L^0 , L^1 or L^∞ when referring to $L^0(\mathcal{F}_T)$, $L^1(\mathcal{F}_T)$ or $L^\infty(\mathcal{F}_T)$, respectively. Whenever we say that the BSDE (ξ, g) has a solution (Y, Z) , we mean that there exists $(Y, Z) \in \mathcal{A}(\xi, g, z)$ such that (2.2) is satisfied with equalities instead of inequalities. Observe that $\mathcal{E}_0^g(\cdot)$, being convex and L^1 -lower semicontinuous, is in particular $\sigma(L^1, L^\infty)$ -lower semicontinuous, and thus, by classical duality results admits the Fenchel-Moreau representation

$$\mathcal{E}_0^g(\xi) = \sup_{v \in L^\infty} \{E[v\xi] - \mathcal{E}_0^*(v)\}, \quad \xi \in L^1, \quad (2.21)$$

where for $v \in L^\infty$ the convex conjugate is given by

$$\mathcal{E}_0^*(v) := \sup_{\xi \in L^1} \{E[v\xi] - \mathcal{E}_0^g(\xi)\}.$$

It is proved in the next lemma that the domain of \mathcal{E}_0^* is concentrated on non-negative $v \in L_+^\infty$ satisfying $E[v] = 1$.

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Lemma 2.18. *Within the representation (2.21), that is $\mathcal{E}_0^g(\xi) = \sup_{v \in L^\infty} \{E[v\xi] - \mathcal{E}_0^*(v)\}$, the supremum might be restricted to those $v \in L_+^\infty$ satisfying $E[v] = 1$.*

Proof. First, we assume without loss of generality that $\mathcal{E}_0^g(0) < +\infty$. Indeed, a slight modification of the argumentation below remains valid using any $\xi \in L^1$ such that $\mathcal{E}_0^g(\xi) < +\infty$.⁴ We show that $\mathcal{E}_0^*(v) = +\infty$ as soon as $v \in L^\infty \setminus L_+^\infty$ or $E[v] \neq 1$. First, for $v \in L^\infty \setminus L_+^\infty$, L_+^1 being the polar of L_+^∞ yields the existence of $\bar{\xi} \in L_+^1$ such that $E[v\bar{\xi}] < 0$. Monotonicity of \mathcal{E}_0^g then gives $\mathcal{E}_0^g(-n\bar{\xi}) \leq \mathcal{E}_0^g(0)$ for all $n \in \mathbb{N}$. Hence,

$$\mathcal{E}_0^*(v) \geq \sup_n \{nE[-v\bar{\xi}] - \mathcal{E}_0^g(-n\bar{\xi})\} \geq \sup_n \{nE[-v\bar{\xi}]\} - \mathcal{E}_0^g(0) = +\infty.$$

Furthermore, since the generator does not depend on y , the function \mathcal{E}_0^g is cash additive, compare Drapeau et al. [24, Proposition 3.3.5], and we deduce that, for all $n \in \mathbb{N}$ it holds

$$\mathcal{E}_0^*(v) \geq E[vn] - \mathcal{E}_0^g(0) - n = n(E[v] - 1) - \mathcal{E}_0^g(0).$$

Thus, if $E[v] > 1$, then $\mathcal{E}_0^*(v) = +\infty$. A reciprocal argument with $\xi = -n$ finally gives $\mathcal{E}_0^*(v) = +\infty$ whenever $E[v] < 1$. \square

By the previous result, we may use $v \in L_+^\infty$, $E[v] = 1$, in order to define a measure Q that is absolutely continuous with respect to P by setting $\frac{dQ}{dP} := v$. Thereby (2.21) may be reformulated as

$$\mathcal{E}_0^g(\xi) = \sup_{Q \ll P} \{E_Q[\xi] - \mathcal{E}_0^*(Q)\}, \quad \xi \in L^1, \quad (2.22)$$

where

$$\mathcal{E}_0^*(Q) := \sup_{\xi \in L^1} \{E_Q[\xi] - \mathcal{E}_0^g(\xi)\}. \quad (2.23)$$

Note that $\mathcal{A}(\xi, g, z) = \emptyset$ implies $\mathcal{E}_0^g(\xi) = +\infty$ and hence such terminal conditions are irrelevant for the supremum in (2.23). Let us denote by \mathcal{Q} the set of all probability measures equivalent to P with bounded Radon-Nikodym derivative. For each $Q \in \mathcal{Q}$, there exists a progressively measurable process q taking values in $\mathbb{R}^{1 \times d}$ such that for all $t \in [0, T]$

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = \exp \left(\int_0^t q_u dW_u - \frac{1}{2} \int_0^t |q_u|^2 du \right).$$

By Girsanov's theorem, the process $W_t^Q := W_t - \int_0^t q_u du$ is a Q -Brownian motion. The following lemma is a valuable tool regarding the characterization of \mathcal{E}_0^* .

Lemma 2.19. *The supremum in (2.23) can be restricted to random variables $\xi \in L^1$ for which the BSDE with parameters (ξ, g) has a solution with value process starting in $\mathcal{E}_0^g(\xi)$. More precisely, for any $Q \in \mathcal{Q}$ holds*

$$\mathcal{E}_0^*(Q) = \sup_{\xi \in L^1} \{E_Q[\xi] - \mathcal{E}_0^g(\xi) : \text{BSDE}(\xi, g) \text{ has a solution } (Y, Z) \text{ with } Y_0 = \mathcal{E}_0^g(\xi)\}.$$

⁴Note that the case $\mathcal{E}_0^g \equiv +\infty$ on L^1 immediately yields $\mathcal{E}_0^* \equiv -\infty$ on L^∞ and is thus neglected.

Proof. It suffices to show that

$$\begin{aligned} & \mathcal{E}_0^*(Q) \\ & \leq \sup_{\xi \in L^1} \{E_Q[\xi] - \mathcal{E}_0^g(\xi) : \text{BSDE}(\xi, g) \text{ has a solution } (Y, Z) \text{ with } Y_0 = \mathcal{E}_0^g(\xi)\}, \end{aligned} \quad (2.24)$$

since the reverse inequality is satisfied by definition of $\mathcal{E}_0^*(\cdot)$. Consider to this end a terminal condition $\xi \in L^1$ with associated minimal supersolution $(Y, Z) \in \mathcal{A}(\xi, g, z)$, that is $Y_0 = \mathcal{E}_0^g(\xi)$. Put, for all $t \in [0, T]$,

$$Y_t^1 = \mathcal{E}_0^g(\xi) - \int_0^t g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^t Z_u dW_u.$$

Relation (2.2) implies $Y_T^1 \geq Y_T \geq \xi$ and thus $\mathcal{E}_0^g(Y_T^1) \geq \mathcal{E}_0^g(\xi)$ and $(Y_T^1)^- \in L^1$. Furthermore, observe that

$$(Y_T^1)^+ = \left(\mathcal{E}_0^g(\xi) - \int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^T Z_u dW_u \right)^+ \leq \left(\mathcal{E}_0^g(\xi) + \int_0^T Z_u dW_u \right)^+$$

due to the positivity of the generator. But since the right-hand side is in L^1 by means of the martingale property of $\int Z dW$, we deduce that $(Y_T^1)^+ \in L^1$, allowing us to conclude that $Y_T^1 \in L^1$. On the other hand, $(Y^1, Z) \in \mathcal{A}(Y_T^1, g, z)$ holds by definition of Y^1 . Hence, we conclude $\mathcal{E}_0^g(Y_T^1) \leq Y_0^1 = \mathcal{E}_0^g(\xi)$. Thus, $\mathcal{E}_0^g(Y_T^1) = \mathcal{E}_0^g(\xi)$, and (Y^1, Z) is a solution of the BSDE with parameters (Y_T^1, g) . Observe further that $\mathcal{E}_0^g(Y_T^1) - \mathcal{E}_0^g(\xi) = 0 \leq Y_T^1 - \xi$ which, by taking expectation under Q , implies

$$E_Q[\xi] - \mathcal{E}_0^g(\xi) \leq E_Q[Y_T^1] - \mathcal{E}_0^g(Y_T^1).$$

Taking the supremum yields (2.24), the proof is done. \square

By means of the preceding lemma it holds

$$\begin{aligned} \mathcal{E}_0^*(Q) &= \sup_{\xi \in L^1} \{E_Q[\xi] - \mathcal{E}_0^g(\xi)\} \\ &= \sup_{\xi \in L^1} \left\{ E_Q \left[\mathcal{E}_0^g(\xi) - \int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^T Z_u dW_u \right] - \mathcal{E}_0^g(\xi) \right\} \\ &= \sup_{(\Delta, \Gamma) \in \Pi} \left\{ E_Q \left[- \int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^T Z_u dW_u \right] \right\} \end{aligned} \quad (2.25)$$

where

$$\Pi := \left\{ (\Delta, \Gamma) \in \mathcal{L}^2 \times \mathcal{L}^2 : \begin{array}{l} \exists \xi \in L^1 : \text{BSDE}(\xi, g) \text{ has a solution } (Y, Z) \\ \text{with } Y_0 = \mathcal{E}_0^g(\xi) \text{ and } Z = z + \int \Delta du + \int \Gamma dW \end{array} \right\}. \quad (2.26)$$

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Whenever $Q \in \mathcal{Q}$, Girsanov's theorem applies and we may exploit the decomposition of Z and use that $\int Z dW^Q$ and $\int \Gamma dW^Q$ are Q -martingales in order to express the right-hand side of (2.25) without Brownian integrals. More precisely,

$$\begin{aligned} \mathcal{E}_0^*(Q) = \sup_{(\Delta, \Gamma) \in \Pi} & \left\{ E_Q \left[\int_0^T \left(-g_u(Z_u, \Delta_u, \Gamma_u) + q_u \int_0^u (\Delta_s + q_s \Gamma_s) ds \right) du \right] \right\} \\ & + z E_Q \left[\int_0^T q_u du \right]. \end{aligned} \quad (2.27)$$

We continue with two lemmata that allow us to restrict the set of measures in the representation (2.22) to a sufficiently nice subset of \mathcal{Q} on the one hand, and to change the set Π appearing in (2.25) to the whole space $\mathcal{L}^2 \times \mathcal{L}^2$ on the other hand.

Lemma 2.20. *Assume there exists some $\xi \in L^1$ such that $\mathcal{A}(\xi, g, z) \neq \emptyset$. Then it is sufficient to consider measures with densities that are bounded away from zero, that is*

$$\mathcal{E}_0^g(\xi) = \sup_{v \in L_b^\infty} \{E[v\xi] - \mathcal{E}_0^*(v)\} \quad (2.28)$$

where $L_b^\infty := \{v \in L^\infty : v > 0 \text{ and } \|\frac{1}{v}\|_{L^\infty} < \infty\}$.

Proof. The assumption of $\mathcal{A}(\xi, g, z)$ being non-empty for some $\xi \in L^1$ implies the existence of $(\Delta, \Gamma) \in \Pi$ and corresponding Z such that $E_P[\int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du] < \infty$ which together with (POS), (2.25) and the martingale property of all occurring $\int Z dW$ under P immediately yields that $\mathcal{E}_0^*(P) < \infty$. For any $Q \ll P$ with $\frac{dQ}{dP} = v \in L_+^\infty$ and $\lambda \in (0, 1)$ we define a measure Q^λ by its Radon-Nikodym derivative $v_\lambda := (1 - \lambda)v + \lambda$ where naturally $\frac{dP}{dQ^\lambda} = 1$. Observe that $\lambda > 0$ implies $v_\lambda \in L_b^\infty$. Next, we show that $\lim_{\lambda \downarrow 0} \mathcal{E}_0^*(v_\lambda) = \mathcal{E}_0^*(v)$. Indeed, convexity of $\mathcal{E}_0^*(\cdot)$ together with $\mathcal{E}_0^*(P) = \mathcal{E}_0^*(1) < \infty$ yields $\liminf_{\lambda \downarrow 0} \mathcal{E}_0^*(v_\lambda) \leq \mathcal{E}_0^*(v)$, whereas the reverse inequality is satisfied by means of the lower semicontinuity. On the other hand, dominated convergence gives $\lim_{\lambda \downarrow 0} E[v_\lambda \xi] = E[v\xi]$, since $|v_\lambda \xi| \leq |v\xi| + |\xi|$ which is integrable. Consequently, the expression $\{E[v\xi] - \mathcal{E}_0^*(v)\}$ is the limit of a sequence $(E[v_{\lambda_n} \xi] - \mathcal{E}_0^*(v_{\lambda_n}))_n$ where $(v_{\lambda_n}) \subset L_b^\infty$ and $\lambda_n \downarrow 0$. Since $\mathcal{E}_0^g(\xi)$ can be expressed as the supremum of $\{E[v\xi] - \mathcal{E}_0^*(v)\}$ over all v , it suffices to consider the supremum over $v \in L_b^\infty$, the proof is done. \square

Lemma 2.21. *For each $Q \in \mathcal{Q}$ such that $\frac{dQ}{dP} \in L_b^\infty$ it holds*

$$\mathcal{E}_0^*(Q) = \sup_{(\Delta, \Gamma) \in \mathcal{L}^2 \times \mathcal{L}^2} \left\{ E_Q \left[- \int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^T Z_u dW_u \right] \right\}. \quad (2.29)$$

Proof. Since Π defined in (2.26) is a subset of $\mathcal{L}^2 \times \mathcal{L}^2$, “ \leq ” certainly holds in (2.29). As to the reverse inequality, observe first that, since we consider a supremum in (2.29) and

$Z \in \mathcal{L}^2$ whenever $(\Delta, \Gamma) \in \mathcal{L}^2 \times \mathcal{L}^2$, those (Δ, Γ) such that $E_Q[\int_0^T g_u(Z_u, \Delta_u, \Gamma_u)du] = +\infty$ can be neglected in the following. In particular, since $\frac{dQ}{dP} = v$ is an element of L_b^∞ , we can restrict our focus to those elements satisfying $E[\int_0^T g_u(Z_u, \Delta_u, \Gamma_u)du] \leq \|\frac{1}{v}\|_{L^\infty} E_Q[\int_0^T g_u(Z_u, \Delta_u, \Gamma_u)du] < +\infty$. Thus, given such a pair (Δ, Γ) , the terminal condition $\xi := -\int_0^T g_u(Z_u, \Delta_u, \Gamma_u)du + \int_0^T Z_u dW_u$ fulfills $\xi^- \in L^1$ due to the martingale property of $\int Z dW$. Furthermore, the pair $(-\int_0^T g_u(Z_u, \Delta_u, \Gamma_u)du + \int_0^T Z_u dW_u, Z)$ is an element of $\mathcal{A}(\xi, g, z)$ by construction and hence Theorem 2.8 yields the existence of $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g, z)$ satisfying $\bar{Y}_0 = \mathcal{E}_0^g(\xi) \leq 0$. Now, using the same techniques as in the proof of Lemma 2.19, we define Y^1 by $Y_t^1 := \mathcal{E}_0^g(\xi) - \int_0^t g_u(\bar{Z}_u, \bar{\Delta}_u, \bar{\Gamma}_u)du + \int_0^t \bar{Z}_u dW_u$, for all $t \in [0, T]$, where $(\bar{\Delta}, \bar{\Gamma})$ is the decomposition of \bar{Z} , and obtain that $Y_T^1 \geq \xi$ as well as $\mathcal{E}_0^g(Y_T^1) = \mathcal{E}_0^g(\xi)$. Consequently,

$$\begin{aligned}
 & -\int_0^T g_u(\bar{Z}_u, \bar{\Delta}_u, \bar{\Gamma}_u)du + \int_0^T \bar{Z}_u dW_u \\
 & = Y_T^1 - \mathcal{E}_0^g(\xi) \geq Y_T^1 \geq \xi = -\int_0^T g_u(Z_u, \Delta_u, \Gamma_u)du + \int_0^T Z_u dW_u, \quad (2.30)
 \end{aligned}$$

which, by taking expectation under Q in (2.30) and using $(\bar{\Delta}, \bar{\Gamma}) \in \Pi$, implies

$$\mathcal{E}_0^*(Q) \geq E_Q \left[-\int_0^T g_u(Z_u, \Delta_u, \Gamma_u)du + \int_0^T Z_u dW_u \right]. \quad (2.31)$$

Since (Δ, Γ) was arbitrary, we have finally shown that $\mathcal{E}_0^*(Q)$ is greater or equal to the supremum over $(\Delta, \Gamma) \in \mathcal{L}^2 \times \mathcal{L}^2$ of the right-hand side of (2.31), which finishes the proof. \square

The ensuing proposition provides, for a given measure $Q \in \mathcal{Q}$ with $\frac{dQ}{dP} \in L_b^\infty$, the existence of a pair of processes attaining the supremum in (2.25).

Proposition 2.22. *For each $Q \in \mathcal{Q}$ with $\frac{dQ}{dP} \in L_b^\infty$ there exist $(\Delta^Q, \Gamma^Q) \in \Pi$ and a corresponding control Z^Q of the form $Z^Q = z + \int \Delta^Q du + \int \Gamma^Q dW$ such that*

$$\mathcal{E}_0^*(Q) = E_Q \left[-\int_0^T g_u(Z_u^Q, \Delta_u^Q, \Gamma_u^Q)du + \int_0^T Z_u^Q dW_u \right]. \quad (2.32)$$

Furthermore, if the convexity of g is strict, then the triple $(Z^Q, \Delta^Q, \Gamma^Q)$ is unique.

Proof. Step 1: The integral $\int q dW$ is an element of BMO. We begin by proving that, for $Q \in \mathcal{Q}$ the density $\frac{dQ}{dP} = \exp(\int_0^T q_u dW_u - \frac{1}{2} \int_0^T |q_u|^2 du)$ of which belongs to L_b^∞ , the process $(\int_0^t q_u dW_u)_{t \in [0, T]}$ is an element of BMO. Indeed, since the process $v_t := E[\frac{dQ}{dP} | \mathcal{F}_t]$ is uniformly bounded away from zero, it satisfies the Muckenaupt

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(A_1) condition, see Kazamaki [48, Definition 2.2], and therefore $\int qdW \in BMO$ by means of Kazamaki [48, Theorem 2.4].

Step 2: \mathcal{L}^2 -boundedness of a minimizing sequence and the candidate (Δ^Q, Γ^Q) . Since the generator g satisfies (DGC), it holds for all (Δ, Γ, Z) that

$$\|\Delta\|_{\mathcal{L}^2(Q)}^2 + \|\Gamma\|_{\mathcal{L}^2(Q)}^2 \leq \frac{1}{c_2} \left(E_Q \left[\int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du \right] - c_1 T \right). \quad (2.33)$$

If we put $F(Z, \Delta, \Gamma) := E_Q[\int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du - \int_0^T q_u \int_0^u (\Delta_s + q_s \Gamma_s) ds du]$, then (2.27) in combination with Lemma 2.21 implies that the conjugate can be expressed by $\mathcal{E}_0^*(Q) = -\inf_{(\Delta, \Gamma) \in \mathcal{L}^2 \times \mathcal{L}^2} F(Z, \Delta, \Gamma) + z E_Q[\int_0^T q_u du]$. We claim that, for $(Z^n, \Delta^n, \Gamma^n)$ a minimizing sequence of F , both (Δ^n) and (Γ^n) are bounded in $\mathcal{L}^2(Q)$. Since in our case the \mathcal{L}^2 -norms with respect to P and Q are equivalent, we suppress the dependence on the measure in the notation to follow. Assume now contrary to our assertion that $\|\Delta^n\|_{\mathcal{L}^2}^2 \rightarrow \infty$ and $\|\Gamma^n\|_{\mathcal{L}^2}^2 \rightarrow \infty$ as n tends to infinity. This in turn would imply either

$$E_Q \left[\int_0^T q_u \int_0^u \Delta_s^n ds du \right] \rightarrow \infty \quad \text{and} \quad \limsup_n \frac{\|\Delta^n\|_{\mathcal{L}^2}^2}{E_Q \left[\int_0^T q_u \int_0^u \Delta_s^n ds du \right]} = K \quad (2.34)$$

or

$$E_Q \left[\int_0^T q_u \int_0^u q_s \Gamma_s^n ds du \right] \rightarrow \infty \quad \text{and} \quad \limsup_n \frac{\|\Gamma^n\|_{\mathcal{L}^2}^2}{E_Q \left[\int_0^T q_u \int_0^u q_s \Gamma_s^n ds du \right]} = L \quad (2.35)$$

or both, for $K, L \in \mathbb{R}$. Indeed, (2.33) would otherwise lead to $\lim_n F(Z^n, \Delta^n, \Gamma^n) = \infty$ and thereby contradict $((Z^n, \Delta^n, \Gamma^n))$ being a minimizing sequence of F . On the other hand however, an application of Hölders inequality yields

$$\begin{aligned} \left| E_Q \left[\int_0^T q_u \int_0^u \Delta_s^n ds du \right] \right| &\leq \left(E_Q \left[\int_0^T |q_u|^2 du \right] E_Q \left[\int_0^T \left(\int_0^u \Delta_s^n ds \right)^2 du \right] \right)^{\frac{1}{2}} \\ &\leq \|q\|_{\mathcal{L}^2} \left(E_Q \left[\int_0^T \int_0^u |\Delta_s^n|^2 ds du \right] \right)^{\frac{1}{2}} \leq T^{\frac{1}{2}} \|q\|_{\mathcal{L}^2} \|\Delta^n\|_{\mathcal{L}^2}. \end{aligned} \quad (2.36)$$

Taking the square on both sides above we obtain

$$\left(E_Q \left[\int_0^T q_u \int_0^u \Delta_s^n ds du \right] \right)^2 \leq T \|q\|_{\mathcal{L}^2}^2 \|\Delta^n\|_{\mathcal{L}^2}^2$$

which in turn implies $\|\Delta^n\|_{\mathcal{L}^2}^2 (E_Q[\int_0^T q_u \int_0^u \Delta_s^n ds du])^{-1} \rightarrow \infty$, a contradiction to (2.34). As to (Γ^n) , we argue similarly and, for $(Q_u)_{u \in [0, T]}$ defined by $Q_u := \int_u^T q_s ds$ estimate

$$\begin{aligned} \left| E_Q \left[\int_0^T q_u \int_0^u q_s \Gamma_s^n ds du \right] \right| &= \left| E_Q \left[\int_0^T q_u \Gamma_u^n Q_u du \right] \right| \leq E_Q \left[\int_0^T |q_u| |\Gamma_u^n| Q_u du \right] \\ &\leq \left(E_Q \left[\int_0^T |q_u|^2 |Q_u|^2 du \right] \right)^{\frac{1}{2}} \|\Gamma^n\|_{\mathcal{L}^2} = \left(E_Q \left[\int_0^T |q_u|^2 \left| \int_u^T q_s ds \right|^2 du \right] \right)^{\frac{1}{2}} \|\Gamma^n\|_{\mathcal{L}^2} \\ &\leq \left(E_Q \left[\left(\int_0^T |q_u|^2 du \right)^2 \right] \right)^{\frac{1}{2}} \|\Gamma^n\|_{\mathcal{L}^2} = \|q\|_{\mathcal{L}^4}^2 \|\Gamma^n\|_{\mathcal{L}^2}, \end{aligned}$$

where we used Fubini's theorem in the first equality above. Since $\int q dW \in BMO$, the \mathcal{L}^4 -norm of q is finite⁵ and the contradiction to (2.35) is derived analogously to the argumentation following (2.36). Consequently, there exists a sequence $((\tilde{\Delta}^n, \tilde{\Gamma}^n))$ in the asymptotic convex hull of $((\Delta^n, \Gamma^n))$ and $(\Delta^Q, \Gamma^Q) \in \mathcal{L}^2 \times \mathcal{L}^2$ such that $((\tilde{\Delta}^n, \tilde{\Gamma}^n))$ converges in $\mathcal{L}^2 \times \mathcal{L}^2$ to (Δ^Q, Γ^Q) . On the side of (Z^n) we pass to the corresponding sequence (\tilde{Z}^n) and recall from the proof of Lemma 2.4 that it is bounded in \mathcal{L}^2 . Hence, there is a sequence in the asymptotic convex hull of (\tilde{Z}^n) , denoted likewise, that converges in \mathcal{L}^2 to some $Z^Q = z + \int \Delta^Q du + \int \Gamma^Q dW$. Of course, we pass the corresponding sequence on the side of $((\tilde{\Delta}^n, \tilde{\Gamma}^n))$ without violating the convergence to (Δ^Q, Γ^Q) .

Step 3: Lower Semicontinuity and convexity of F . In a next step we show that the earlier defined function $F(Z, \Delta, \Gamma) = E_Q[\int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du - \int_0^T Z_u dW_u]$ is lower semicontinuous and convex on $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{L}^2$ where $Z = z + \int \Delta du + \int \Gamma dW$. Indeed, the part $E_Q[\int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du]$ is lower semicontinuous by (POS), (LSC) and Fatou's lemma. As to the second part, first observe that \mathcal{L}^2 -convergence of (\tilde{Z}^n) towards Z^Q implies

$$\left| E_Q \left[\int_0^T (\tilde{Z}_u^n - Z_u^Q) dW_u \right] \right| \xrightarrow{n \rightarrow \infty} 0.$$

Furthermore, (CON) yields that F is convex in (Z, Δ, Γ) .

Step 4: Minimality of $(Z^Q, \Delta^Q, \Gamma^Q)$. We claim that $F(Z^Q, \Delta^Q, \Gamma^Q) = \inf_{(\Delta, \Gamma) \in \mathcal{L}^2 \times \mathcal{L}^2} F(Z, \Delta, \Gamma)$ which would then in turn finally imply (2.32). To this end, it suffices to prove that $F(Z^Q, \Delta^Q, \Gamma^Q) \leq \inf_{(\Delta, \Gamma) \in \mathcal{L}^2 \times \mathcal{L}^2} F(Z, \Delta, \Gamma)$, since the reverse inequality is

⁵Recall that BMO can be embedded into any \mathcal{H}^p -space, compare Kazamaki [48, Section 2.1, p. 26].

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naturally satisfied. Observe now that

$$\begin{aligned} \inf_{(\Delta, \Gamma) \in \mathcal{L}^2 \times \mathcal{L}^2} F(Z, \Delta, \Gamma) &= \lim_n F(Z^n, \Delta^n, \Gamma^n) = \lim_n \sum_{k=n}^{M(n)} \lambda_k^{(n)} F(Z^k, \Delta^k, \Gamma^k) \\ &\geq \lim_n F\left(\sum_{k=n}^{M(n)} \lambda_k^{(n)} Z^k, \sum_{k=n}^{M(n)} \lambda_k^{(n)} \Delta^k, \sum_{k=n}^{M(n)} \lambda_k^{(n)} \Gamma^k\right) \\ &= \lim_n F(\tilde{Z}^n, \tilde{\Delta}^n, \tilde{\Gamma}^n) \geq F(Z^Q, \Delta^Q, \Gamma^Q) \end{aligned}$$

where we denoted by $\lambda_k^{(n)}$, $n \leq k \leq M(n)$, $\sum_k \lambda_k^{(n)} = 1$ the convex weights of the sequence $((\tilde{Z}^n, \tilde{\Delta}^n, \tilde{\Gamma}^n))$ and made use of the convexity and lower semicontinuity of F .

Step 5: Uniqueness of $(Z^Q, \Delta^Q, \Gamma^Q)$. As to the uniqueness, assume that there are (Δ^1, Γ^1) and (Δ^2, Γ^2) with corresponding Z^1 and Z^2 , respectively, both attaining the supremum such that $P \otimes dt[(\Delta^1, \Gamma^1) \neq (\Delta^2, \Gamma^2)] > 0$. Setting $(\bar{Z}, \bar{\Delta}, \bar{\Gamma}) := \frac{1}{2}[(Z^1, \Delta^1, \Gamma^1) + (Z^2, \Delta^2, \Gamma^2)]$ together with $Q \sim P$ and the strict convexity of F inherited by g yields that $F(\bar{Z}, \bar{\Delta}, \bar{\Gamma}) < F(Z^1, \Delta^1, \Gamma^1)$, a contradiction to the optimality of $(Z^1, \Delta^1, \Gamma^1)$. \square

Remark 2.23. Since for a given $Q \in \mathcal{Q}$ with $\frac{dQ}{dP} \in L_b^\infty$ and a strictly convex generator the maximizer $(Z^Q, \Delta^Q, \Gamma^Q)$ is unique by the preceding proposition, it has to be (conditionally) optimal at all times $t \in [0, T]$. Indeed, assume to the contrary the existence of (Δ, Γ) such that $z + \int_0^t \Delta_u du + \int_0^t \Gamma_u dW_u = Z_t = Z_t^Q$ and $E_Q[\int_t^T -g_u(Z_u, \Delta_u, \Gamma_u) du + \int_t^T Z_u dW_u | \mathcal{F}_t] > E_Q[\int_t^T -g_u(Z_u^Q, \Delta_u^Q, \Gamma_u^Q) du + \int_t^T Z_u^Q dW_u | \mathcal{F}_t]$ holds true for some $t \in [0, T]$. Then, however, the concatenated triple $(\bar{Z}, \bar{\Delta}, \bar{\Gamma}) := (Z^Q, \Delta^Q, \Gamma^Q)1_{[0, t]} + (Z, \Delta, \Gamma)1_{[t, T]}$ satisfies $E_Q[\int_0^T -g_u(\bar{Z}_u, \bar{\Delta}_u, \bar{\Gamma}_u) du + \int_0^T \bar{Z}_u dW_u] > E_Q[\int_0^T -g_u(Z_u^Q, \Delta_u^Q, \Gamma_u^Q) du + \int_0^T Z_u^Q dW_u]$, a contradiction to the optimality of $(Z^Q, \Delta^Q, \Gamma^Q)$ at time zero. \blacklozenge

Notice that, for $d = 1$ and the case of a quadratic generator which is in addition independent of z , that is $g_u(\delta, \gamma) = |\delta|^2 + |\gamma|^2$, the processes (Δ^Q, Γ^Q) attaining $\mathcal{E}_0^*(Q)$ can be explicitly computed and (Δ_t^Q, Γ_t^Q) depends on the whole path of q up to time t , as illustrated in the following proposition. It thus constitutes a useful tool for the characterization of the dual optimizers and its proof is closely related to the Euler-Lagrange equation arising in classical calculus of variation.

Proposition 2.24. *Assume that $d = 1$ and that g is defined by $g_u(\delta, \gamma) = |\delta|^2 + |\gamma|^2$. For $Q \in \mathcal{Q}$ with $\frac{dQ}{dP} \in L_b^\infty$, let (Δ^Q, Γ^Q) be the optimizer attaining $\mathcal{E}_0^*(Q)$. Then there*

exist $c_1, c_2 \in \mathbb{R}$ such that

$$\Delta_t^Q = -\frac{1}{2} \int_0^t q_s ds + c_1 \quad (2.37)$$

$$\Gamma_t^Q = -\frac{1}{2} q_t \left(\int_0^t q_s ds + c_2 \right), \quad (2.38)$$

for all $t \in [0, T]$.

Proof. For the purpose of this proof we assume without loss of generality that $z = 0$, since the initial value does not affect the optimization with respect to (Δ, Γ) . Hence, the generator only depending on δ and γ in combination with (2.27) gives $\mathcal{E}_0^*(Q) = -\inf_{(\Delta, \Gamma)} \{F_1(\Delta) + F_2(\Gamma)\}$ where $F_1(\Delta) := E_Q[\int_0^T (|\Delta_u|^2 - q_u \int_0^u \Delta_s ds) du]$ and $F_2(\Gamma) := E_Q[\int_0^T (|\Gamma_u|^2 - q_u \int_0^u q_s \Gamma_s ds) du]$. We will proceed along an ω -wise criterion of optimality, since any pair (Δ^Q, Γ^Q) that is optimal for almost all $\omega \in \Omega$ then naturally also optimizes the expectation under Q . The uniqueness obtained in Proposition 2.22 then assures that the path-wise optimizer is the only one. We define

$$J_1(\Delta) = \int_0^T \left(|\Delta_u|^2 - q_u \int_0^u \Delta_s ds \right) du \quad \text{and} \quad J_2(\Gamma) = \int_0^T \left(|\Gamma_u|^2 - q_u \int_0^u q_s \Gamma_s ds \right) du$$

and observe that it is sufficient to elaborate how to obtain conditions for a minimizer of J_1 , as the functional J_2 is of a similar structure. Introducing $X(u) := \int_0^u \Delta_s ds$ we obtain $X'(u) := \frac{d}{du} X(u) = \Delta_u$ and

$$J_1(\Delta) = \tilde{J}_1(X) = \int_0^T L(u, X(u), X'(u)) du$$

where $L(u, a, b) = |b|^2 - q_u a$. If X is a local minimum of \tilde{J}_1 , then $\tilde{J}_1(X) \leq \tilde{J}_1(X + \varepsilon \eta)$ for sufficiently small $\varepsilon > 0$ and all differentiable $\eta \in C([0, T], \mathbb{R})$ the derivatives of which are square integrable and which satisfy $\eta(0) = \eta(T) = 0$. In particular, with $\phi(\varepsilon) := \tilde{J}_1(X + \varepsilon \eta)$, it has to hold that $\frac{d}{d\varepsilon} \phi(\varepsilon)|_{\varepsilon=0} = 0$. Using the specific form of L we get

$$\begin{aligned} \frac{d}{d\varepsilon} \phi(\varepsilon)|_{\varepsilon=0} &= \lim_{h \rightarrow 0} \frac{1}{h} \int_0^T [L(u, X(u) + h\eta(u), X'(u) + h\eta'(u)) - L(u, X(u), X'(u))] du \\ &= \lim_{h \rightarrow 0} \int_0^T [-q_u \eta(u) + 2X'(u)\eta'(u) + h(\eta'(u))^2] du. \end{aligned}$$

Having assumed η' to be square integrable allows us to exchange limit and integration,

2. Minimal Supersolutions of Convex BSDEs under Constraints – Existence and Duality

yielding

$$0 = \frac{d}{d\varepsilon} \phi(\varepsilon)|_{\varepsilon=0} = \int_0^T [-q_u \eta(u) + 2\Delta_u \eta'(u)] du. \quad (2.39)$$

Using integration by parts we obtain

$$-\int_0^T q_u \eta(u) du = \left(\int_0^u -q_s ds \right) \eta(u) \Big|_0^T - \int_0^T \left(\int_0^u -q_s ds \right) \eta'(u) du.$$

The first term on the right-hand side above vanishes and so, by plugging this back into (2.39) we end up with

$$\int_0^T \left(2\Delta_u + \int_0^u q_s ds \right) \eta'(u) du = 0. \quad (2.40)$$

Let us next introduce the constant $c := \frac{1}{T} \int_0^T (2\Delta_u + \int_0^u q_s ds) du$ and observe that, using $\int_0^T \eta'(u) du = 0$, Equation (2.40) may be rewritten as

$$\int_0^T \left(2\Delta_u + \int_0^u q_s ds - c \right) \eta'(u) du = 0. \quad (2.41)$$

Moreover, the function

$$\bar{\eta}(t) := \int_0^t \left(2\Delta_u + \int_0^u q_s ds - c \right) du$$

satisfies $\bar{\eta}(0) = \bar{\eta}(T) = 0$ by construction as well as $\bar{\eta}'(u) = 2\Delta_u + \int_0^u q_s ds - c$, which is square integrable for almost all $\omega \in \Omega$, since Δ and q are square integrable⁶. Hence, (2.41) applied to our particular function $\bar{\eta}$ yields

$$\int_0^T \left(2\Delta_u + \int_0^u q_s ds - c \right)^2 du = 0$$

and we deduce that

$$2\Delta_u + \int_0^u q_s ds = \frac{1}{T} \int_0^T \left(2\Delta_r + \int_0^r q_s ds \right) dr \quad \text{for almost all } u \in [0, T]. \quad (2.42)$$

Specifically, (2.42) shows that, for almost all $\omega \in \Omega$, there exists a set $\mathcal{I}(\omega) \subseteq [0, T]$ with

⁶More precisely, it holds $P(\int_0^T |q_u|^2 du < \infty) = P(\int_0^T |\Delta_u|^2 du < \infty) = 1$.

2.3. Duality under Constraints

Lebesgue measure T such that, for all $u \in \mathcal{I}(\omega)$,

$$2\Delta_u + \int_0^u q_s ds = M, \quad (2.43)$$

where M of course depends on $\omega \in \Omega$ and is thus a random variable. This in turn implies, for dt -almost all $u \in [0, T]$, the existence of $\Omega_u \subseteq \Omega$ with $P(\Omega_u) = 1$ such that (2.43) holds for all $\omega \in \Omega_u$. In particular, on Ω_u the above M equals an \mathcal{F}_u -measurable random variable. We choose a sequence $(u_n) \subset [0, T]$ with $\lim_n u_n = 0$ and hence obtain that on $\bar{\Omega} := \bigcap_n \Omega_{u_n}$, where $P(\bar{\Omega}) = Q(\bar{\Omega}) = 1$, M equals an $\bigcap_n \mathcal{F}_{u_n}$ -measurable random variable and is thus \mathcal{F}_0 -measurable, that means it is a constant on $\bar{\Omega}$, by the right-continuity of our filtration. Since modifying our optimizer on a Q -nullset does not alter the value of the functional to be optimized, we have shown (2.37) by putting $c_1 := \frac{M}{2}$.

As to the case of our optimal Γ^Q , assume first that $q_u \neq 0$ for all $u \in [0, T]$ and observe that, with $Y(u) := \int_0^u q_s \Gamma_s ds$, we obtain $J_2(\Gamma) = \tilde{J}_2(Y) = \int_0^T K(u, Y(u), Y'(u)) du$ where we set $K(u, a, b) = (1/q_u)^2 b^2 - q_u a$. Thus, an argumentation identical to that above would yield (2.38) in that case. Furthermore, it holds that $\lim_{q_u \rightarrow 0} \Gamma_u^Q = 0$, a value that is consistent with the “pointwise” minimization consideration that $\arg \min_{\Gamma_u} \{|\Gamma_u|^2 - q_u \int_{[0, u]} q_s \Gamma_s ds\}|_{q_u=0} = 0$ and hence justifies expression (2.38). \square

For $\xi \in L^1$ such that $\mathcal{A}(\xi, g, z) \neq \emptyset$, the ensuing theorem states that, given the existence of an equivalent probability measure $\hat{Q} \in \mathcal{Q}$ such that the sup in (2.22) is attained, the BSDE with generator g and terminal condition ξ admits a solution under constraints.

Theorem 2.25. *Assume that, for $\xi \in L^1$ with $\mathcal{A}(\xi, g, z) \neq \emptyset$, there exists a $\hat{Q} \in \mathcal{Q}$ with $\frac{dQ}{dP} \in L_b^\infty$ such that $\mathcal{E}_0^g(\xi) = E_{\hat{Q}}[\xi] - \mathcal{E}_0^*(\hat{Q})$. Then there exists a solution $(Y, Z) \in \mathcal{A}(\xi, g, z)$ of the BSDE with parameters (ξ, g) .*

Proof. Starting with (2.22) in combination with Proposition 2.22, it holds

$$\mathcal{E}_0^g(\xi) = E_{\hat{Q}}[\xi] - \mathcal{E}_0^*(\hat{Q}) = E_{\hat{Q}} \left[\xi + \int_0^T g_u(Z_u^{\hat{Q}}, \Delta_u^{\hat{Q}}, \Gamma_u^{\hat{Q}}) du - \int_0^T Z_u^{\hat{Q}} dW_u \right]. \quad (2.44)$$

We recall that $\mathcal{A}(\xi, g, z) \neq \emptyset$ and $\xi \in L^1$. Hence, by Theorem 2.8 there exists (Z, Δ, Γ) such that

$$E_{\hat{Q}} \left[\xi + \int_0^T g_u(Z_u^{\hat{Q}}, \Delta_u^{\hat{Q}}, \Gamma_u^{\hat{Q}}) du - \int_0^T Z_u^{\hat{Q}} dW_u \right] - \int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^T Z_u dW_u \geq \xi$$

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holds true. Taking expectation under \hat{Q} on both sides of the inequality above yields

$$E_{\hat{Q}} \left[- \int_0^T g_u(Z_u^{\hat{Q}}, \Delta_u^{\hat{Q}}, \Gamma_u^{\hat{Q}}) du + \int_0^T Z_u^{\hat{Q}} dW_u \right] \leq E_{\hat{Q}} \left[- \int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^T Z_u dW_u \right].$$

However, the expression on the left-hand side is maximal for $(Z^{\hat{Q}}, \Delta^{\hat{Q}}, \Gamma^{\hat{Q}})$ by means of Proposition 2.22 and thus equality has to hold. Hence, it follows that $\mathcal{E}_0^*(\hat{Q}) = E_{\hat{Q}}[-\int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^T Z_u dW_u]$. By plugging this back into (2.44) we obtain

$$\mathcal{E}_0^g(\xi) = E_{\hat{Q}}[\xi] + E_{\hat{Q}} \left[\int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du - \int_0^T Z_u dW_u \right]$$

which is equivalent to

$$E_{\hat{Q}} \left[\mathcal{E}_0^g(\xi) - \int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^T Z_u dW_u - \xi \right] = 0.$$

Since the expression within the expectation is P - and thereby also \hat{Q} -almost surely positive, we finally conclude that

$$\mathcal{E}_0^g(\xi) - \int_0^T g_u(Z_u, \Delta_u, \Gamma_u) du + \int_0^T Z_u dW_u = \xi,$$

and thus $(\mathcal{E}_0^g(\xi) - \int_0^\cdot g(Z, \Delta, \Gamma) du + \int_0^\cdot Z dW, Z)$ constitutes a solution of the BSDE with parameters (ξ, g) . \square

Part II.

Equilibrium Pricing in Generalized CAPMs

3. Continuous Equilibrium in Affine and Information-Based Capital Asset Pricing Models

3.1. Introduction

In this chapter we propose an analytically-tractable equilibrium model in continuous time, within which financial securities are priced in a generalized capital asset pricing model (CAPM).

It is well known that when markets are incomplete, competitive equilibria may fail to exist. Even if they exist, they may not be Pareto optimal, nor supportable as equilibria of a suitable representative agent economy. The equilibrium analysis of incomplete markets is therefore always confined to special cases, for instance to single agent models (He and Leland [35], Gârleanu et al. [33]), multiple agent models where markets are complete in equilibrium (Duffie and Huang [27], Horst et al. [40], Karatzas et al. [47]), or models with particular classes of goods (Jofre et al. [43]) or preferences (Carmona et al. [12]).

In Cheridito et al. [16], the authors recently established existence and uniqueness of equilibrium results for incomplete financial market models in discrete time when agents' preferences are translation invariant¹. In the situation where uncertainty is spanned by finitely many random walks, they showed that the equilibrium dynamics can be described as the solution to a coupled system of forward-backward stochastic difference equations. The system is usually high-dimensional because one obtains one equation per security and market participant. This renders simulations and calibrations of the model cumbersome, if not impossible. Within the framework of generalized CAPMs, that is, if all agents share the same base preferences (as in the case of exponential utility functions) and the endowments lie in the span of the tradable assets, the system simplifies to a single equation representing the equilibrium utility of some representative agent. Furthermore, the equilibrium price process depends only on the aggregated endowment, the market risk aversion, and the flow of market information. It is in this sense that these three items fully characterize equilibrium prices in generalized CAPMs.

In this chapter we extend the generalized CAPM analyzed in Cheridito et al. [16] to continuous time when agents' preferences are of the expected exponential type. In particular, the advantage of the herewith presented continuous-time framework is that

¹Exponential utility functions, for instance, are translation invariant after a logarithmic transformation.

3.2. A Generalized Capital Asset Pricing Model

we obtain (semi-)explicit formulae for equilibrium prices. If not explicitly computable, key equilibrium quantities can be computed using numerical integration only—no Monte Carlo methods are needed. We consider a model with a finite number of agents, which are initially endowed with an attainable random payoff. They trade a finite number of securities so as to maximize expected exponential utility from terminal wealth. The financial securities are characterized by their terminal payoffs, which we assume to be functions of finitely many market factors. The market factors may or may not be observable to the agents. Affine processes, and the theory of information-based asset pricing are used to model the endogenous asset price dynamics and the terminal payoff.

Within our first approach, the dynamics of the market factors follows an affine process that generates the market filtration. Affine processes are extensively used in mathematical finance (see for instance Duffie and Singleton [28], Duffie et al. [29], Keller-Ressel [49] and references therein), as they lend themselves to a transparent mathematical analysis and to the application of efficient numerical methods. We show that within an affine framework, equilibrium securities prices are given by the quotient of two integrals. Both integrals are the product of an exponential function evaluated at the current state of the factor process and the Fourier transform of a smooth function. Representing equilibrium prices in terms of deterministic integrals allows for a fast and efficient numerical analysis of other equilibrium quantities, such as option implied volatilities. We analyze implied volatilities for two single-security benchmark models: (i) an additive Heston stochastic volatility model, and (ii) a pure jump Ornstein-Uhlenbeck model. Both models reproduce the well-documented smile-effect of implied volatilities and identify investor risk aversion as a key determinant of implied volatilities.

The second approach to continuous equilibrium presented in this chapter is based on the theory of information-based asset pricing, see Brody et al. [11] and Hoyle et al. [42]. Within this approach, the asset price dynamics are explicitly generated by taking the conditional expectation of the future cash flows, which are multiplied by the pricing rule, given the partial information about the market factors that is available to the agents. The filtration is modeled by stochastic processes, which (i) carry information about the a priori distribution of the market factors, and (ii) embody pure noise preventing market participants from accessing full knowledge as to what is the “true” value of the asset at any time before the cash flows occur. We use the information-based framework to show the dependence of the equilibrium prices of credit-risky securities on information about the financial standing of a company.

The chapter is structured as follows. A general existence result along with a discussion on the information-generating processes is given in Section 3.2. In Section 3.3 and 3.4 we present affine and information-based equilibrium pricing models, respectively. A brief addendum to regular affine processes can be found in the appendix.

3.2. A Generalized Capital Asset Pricing Model

We consider an equilibrium model in continuous time with a finite set \mathbb{A} of economic agents. Uncertainty is modeled by a probability space (Ω, \mathcal{F}, P) carrying a filtration

3. Continuous Equilibrium in Affine and Information-Based CAPMs

$(\mathcal{F}_t)_{t \in [0, T]}$. The filtration captures the flow of information that is available to the agents over the trading period $[0, T]$, and is assumed to satisfy the usual assumptions of completion and right-continuity. In what follows, all equalities and inequalities are to be understood in the P -almost sure sense.

3.2.1. Existence of Equilibrium

The agents can lend to and borrow from the money market account at some exogenously given interest rate, and they can trade K securities. The securities are in net supply $n = (n^1, \dots, n^K) \in \mathbb{R}^K$ and characterized by their terminal payoffs $S_T = (S_T^1, \dots, S_T^K)$, which we assume to be \mathcal{F}_T -measurable random variables. Securities are priced to match demand and supply. Each agent $a \in \mathbb{A}$ is initially endowed with some \mathcal{F}_T -measurable random payoff H^a of the form

$$H^a = c^a + \eta^a \cdot S_T,$$

for constants $c^a \in \mathbb{R}$ and $\eta^a \in \mathbb{R}^K$. Furthermore, at each time $t \in [0, T]$ the agent's preferences can be described by the utility functional

$$U_t^a(X) = -\frac{1}{\gamma^a} \log \left(E \left[e^{-\gamma^a X} \mid \mathcal{F}_t \right] \right),$$

where $\gamma^a > 0$ is the risk aversion parameter. Thus at time $t \in [0, T]$, the agent faces the optimization problem

$$\sup_{\vartheta \in \Theta} U_t^a \left(H^a + \int_t^T \vartheta_u dS_u \right),$$

where the set of admissible trading strategies Θ is given by

$$\Theta = \left\{ \vartheta \in L(S) : G(\vartheta) \text{ is a } \tilde{Q}\text{-supermartingale, for all } \tilde{Q} \in \mathcal{P} \right\}.$$

Here, $L(S)$ and $G_t(\vartheta) := \int_0^t \vartheta_u dS_u$ denote the set of S -integrable predictable processes and the gains process, respectively, whereas \mathcal{P} denotes the set of all equivalent martingale measures (EMM) for S .²

The goal is now to establish existence of a (discounted) equilibrium price process $(S_t)_{t \in [0, T]}$.³ Since all agents share the same base preferences, and because all payoffs lie in the span of the tradable assets, our model can be viewed as a generalized CAPM. Just like in the classical CAPM, in our incomplete market model existence of an equilibrium can be established using the standard representative agent approach that underlies equi-

²Note that in equilibrium, there is an EMM Q , that is, an equivalent probability measure Q under which the price process S will be a true martingale. In particular, S will be a P -semimartingale. For related discussions on suitable sets of admissible strategies see for instance Delbaen and Schachermayer [20], Delbaen et al. [21], or Biagini and Frittelli [5].

³For simplicity, we assume that the trading horizon T is short so that interest rate risk can be ignored.

3.2. A Generalized Capital Asset Pricing Model

librium models of complete markets. Furthermore, all agents share the market portfolio according to their risk aversion in equilibrium. The equilibrium pricing kernel depends on the agents' preferences and endowments, however only through the endowment- and supply-adjusted risk aversion

$$\tilde{\gamma} := \gamma(\eta + n) \in \mathbb{R}^K. \quad (3.1)$$

Here, $\eta := \sum_a \eta^a$ denotes the aggregate endowment and $\gamma^{-1} := \sum_a \gamma_a^{-1}$ can be viewed as the market risk aversion. The following result can be proved by standard duality results for entropic utility functions; see Cheridito et al. [16, Theorem 5.1] for a related result in discrete time.

Theorem 3.1. *Suppose that the following integrability conditions hold:*

$$\exp(-\tilde{\gamma} \cdot S_T) \in L^1(P) \quad \text{and} \quad S_T \in L^1(Q)^K, \quad (3.2)$$

where Q is an equivalent probability measure with density

$$\frac{dQ}{dP} = \frac{\exp(-\tilde{\gamma} \cdot S_T)}{E[\exp(-\tilde{\gamma} \cdot S_T)]}. \quad (3.3)$$

Then, the price process S defined by

$$S_t = E_Q[S_T \mid \mathcal{F}_t], \quad t \in [0, T], \quad (3.4)$$

together with the constant trading strategies

$$\hat{\vartheta}_t^a \equiv \frac{\gamma}{\gamma^a}(n + \eta) - \eta^a, \quad a \in \mathbb{A},$$

constitutes an equilibrium.

Proof. Due to the time-consistency and strict monotonicity of the entropic preferences, it suffices to show that the strategies $\hat{\vartheta}^a$ are optimal for the utility maximization in $t = 0$. Note first that (3.2) ensures that (3.3) and (3.4) are well-defined. In particular, the price process S is a Q -martingale, and thus $Q \in \mathcal{P}$. Furthermore, the constant strategies $\hat{\vartheta}^a$ lie in Θ , since for any $\tilde{Q} \in \mathcal{P}$, the process $G_t(\hat{\vartheta}^a) = \hat{\vartheta}^a \cdot (S_t - S_0)$ is by assumption a \tilde{Q} -martingale, and hence in particular a \tilde{Q} -supermartingale.

We now show that the quantity γ introduced in (3.1) can be seen as the risk aversion of some representative agent whose optimal utility is attained at the constant strategy $\vartheta^* \equiv n + \eta$. Indeed, since S is a Q -martingale, and $(n + \eta) \cdot S_T \in L^1(Q)$, the utility

3. Continuous Equilibrium in Affine and Information-Based CAPMs

maximization of the representative agent can be formulated as follows⁴:

$$\begin{aligned}
& \sup_{\vartheta \in \Theta, E_Q[G_T(\vartheta)] \leq E_Q[(n+\eta) \cdot S_T]} \{U_0^\gamma(G_T(\vartheta))\} \\
& \leq \sup_{\vartheta \in \Theta} \{U_0^\gamma(G_T(\vartheta) - E_Q[G_T(\vartheta)] + E_Q[(n+\eta) \cdot S_T])\} \\
& = \sup_{\vartheta \in \Theta} \{U_0^\gamma(G_T(\vartheta)) - E_Q[G_T(\vartheta)]\} + E_Q[(n+\eta) \cdot S_T] \\
& \leq \frac{1}{\gamma} H(Q|P) + E_Q[(n+\eta) \cdot S_T] .
\end{aligned} \tag{3.5}$$

The last inequality is derived from the dual representation of U_0^γ , where the relative entropy is given by $H(Q|P) = E[\frac{dQ}{dP} \log(\frac{dQ}{dP})]$. But $G_T(\vartheta^*)$ with $\vartheta^* \equiv n + \eta$ plugged into the representative agent's utility $U_0^\gamma(\cdot)$ yields

$$U_0^\gamma((n+\eta) \cdot S_T) = \frac{1}{\gamma} H(Q|P) + E_Q[(n+\eta) \cdot S_T] .$$

Comparing this with (3.5) shows that $\vartheta^* \equiv n + \eta$ is indeed optimal for the representative agent when the price process S is given by (3.4). Individual optimality of $\hat{\vartheta}^a$ for the single agents now follows by a scaling argument and the specific form of the aggregated endowment. Note that, for all $a \in \mathbb{A}$,

$$\vartheta^* = \arg \max_{\vartheta \in \Theta} \{U_0^\gamma(G_T(\vartheta))\}$$

is equivalent to

$$\frac{\gamma}{\gamma^a} \vartheta^* = \arg \max_{\vartheta \in \Theta} \{U_0^a(G_T(\vartheta))\} ,$$

which in turn is equivalent to

$$\frac{\gamma}{\gamma^a} \vartheta^* - \eta^a = \arg \max_{\vartheta \in \Theta} \{U_0^a(H^a + G_T(\vartheta))\} .$$

This shows that $\hat{\vartheta}^a$ is the optimal strategy for agent $a \in \mathbb{A}$. Since the strategies $(\hat{\vartheta}^a)_{a \in \mathbb{A}}$ add up to n , the market clears at any time, and hence the pair $((S_t)_{t \in [0, T]}, (\hat{\vartheta}^a)_{a \in \mathbb{A}})$ forms an equilibrium. \square

We notice that the equilibrium pricing kernel Q depends only on the terminal payoffs weighted by the endowment- and supply-adjusted risk aversion. In particular, if the k -th security is in zero endowment-adjusted supply, that is, if $\eta^k + n^k = 0$, then its payoff does not affect the equilibrium pricing kernel.

Furthermore, the integrability assumption on S under the pricing measure Q guarantees that equilibrium prices are Q -martingales. Hence they are, by (3.2) and (3.3), P -semimartingales and thus well defined as an integrator in the sense of Protter [58,

⁴Note that the first expression in (3.5) is equivalent to the representative agent's utility maximization of terminal wealth against both, the aggregated initial endowments η and aggregated net supply n , over all admissible strategies.

Chapter II and IV].

3.2.2. The Market Filtration

The previous theorem established existence of a continuous equilibrium under no assumptions on the underlying filtration (\mathcal{F}_t) . We emphasize that the construction of the filtration characterizes the dynamics of the derived price processes. In order to obtain (semi-)explicit equilibrium price processes, we assume that the terminal payoffs depend in a functional form on a vector X of market factors the distribution of which is known to the agents. We define the following:

$$S_T^k = f^k(X).$$

We assume that the market filtration $(\mathcal{F}_t)_{t \in [0, T]}$, to which the equilibrium prices will be adapted, is generated by an observable stochastic process (ξ_t) such that, possibly up to a constant, $\xi_T = X$. Equilibrium dynamics are then studied within an affine and an information-based framework. The first approach assumes that the dynamics of the market factors follow an affine process; in the second approach the observables generating the market filtration are modeled by Brownian random bridges with drift from zero to X .

3.3. Affine Equilibrium Framework

In this section, we assume that the dynamics of the market factors ξ are observable and that they follow an affine process Y , that is $\xi = Y$. After specifying the setup and following a brief introduction into the theory of affine processes, the results in Section 3.2 are used to derive equilibrium pricing formulae in Section 3.3.1. This is followed by an analysis of equilibrium option prices in Section 3.3.2 and equilibrium asset prices in a Heston stochastic volatility framework and an Ornstein-Uhlenbeck jump model in Sections 3.3.3 and 3.3.4, respectively. Since we consider a linear payoff structure of the underlying asset $S_T = X_T$ from Section 3.3.2 onwards, negative equilibrium prices can not be excluded a priori. However, this can be avoided by either directly modeling the log-payoff of the underlying, that is $S_T = \exp(X_T)$, or, as in the present work, considering only short trading horizons T . In this case, option prices obtained from a model and its “logarithmic counterpart” are quite close, compare for instance the discussion in Schachermayer and Teichmann [61]. We choose the latter approach, since the verification of the integrability conditions in Theorem 3.3 is more involved in the case of a log-payoff. The additional challenge is due to the “double exponential” structure. We emphasize however that, once this is achieved, all our results can be adapted and hence extended also to longer trading horizons.

3.3.1. Setup and Equilibrium Pricing Formulae

In this section, we consider the case where the payoff S_T is a functional of an observable affine factor process. To this end, we assume that the underlying probability space (Ω, \mathcal{F}, P) is rich enough to support an affine Markov process Y taking values in the state space $D := \mathbb{R}_+^m \times \mathbb{R}^n$.

We set $d = m + n$ and write $Y = (V, X)$. We interpret $X \in \mathbb{R}^n$ as the factor process that determines the payoff and $V \in \mathbb{R}_+^m$ as a process driving it; a typical example would be a stochastic volatility model. We assume that Y^T , the Markov process stopped at time T , is conservative, meaning that there are no explosions or absorbing states up to time T . The market filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is then chosen to be the one generated by Y :

$$\mathcal{F}_t = \sigma(Y_s, s \leq t), \quad t \in [0, T].$$

Usually, one associates with Y a family of probability measures $(P^y)_{y \in D}$, which represents the law of the process Y starting at $y \in D$. Since every affine process is a Feller process, the filtration (\mathcal{F}_t) can be completed with respect to the family $(P^y)_{y \in D}$ so that the filtration is automatically right-continuous, compare Revuz and Yor [60, Section III.2].

Affine processes

Before turning to the problem of equilibrium pricing, we recall some useful results on affine processes, the details of which can be found in Duffie et al. [29] or in Keller-Ressel [49].

Definition 3.2. An affine process is a stochastically continuous⁵, time-homogeneous Markov process (Y, P^y) with state-space D , of which log-characteristic function is an affine function of the state vector. That is, there exist functions $\phi : \mathbb{R}_+ \times i\mathbb{R}^d \rightarrow \mathbb{C}$ and $\psi : \mathbb{R}_+ \times i\mathbb{R}^d \rightarrow \mathbb{C}^d$ such that

$$E^y [\exp(u \cdot Y_t)] = \exp[\phi(t, u) + \psi(t, u) \cdot y], \quad (3.6)$$

for all $y \in D$ and $(t, u) \in \mathbb{R}_+ \times i\mathbb{R}^d$. An affine process Y is called regular, if the derivatives

$$F(u) := \partial_t \phi(t, u)|_{t=0+}, \quad R(u) := \partial_t \psi(t, u)|_{t=0+}$$

exist for all $u \in \mathcal{U} := \{u = (u_v, u_x) \in \mathbb{C}^m \times \mathbb{C}^n : \text{Re}(u_v) \leq 0, \text{Re}(u_x) = 0\}$ and are continuous in $u = 0$.⁶

⁵A stochastic process Y is stochastically continuous, if for any sequence $(t_m) \rightarrow t$ in \mathbb{R}_+ , Y_{t_m} converges to Y_t in probability.

⁶In the recent work Keller-Ressel et al. [51], the authors actually show that each affine process as defined above is regular, whereas in Duffie et al. [29] and Keller-Ressel [49] regularity is still an assumption on Y .

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The definition of an affine process Y implies that the \mathcal{F}_t -conditional characteristic function of Y_T ($T \geq t$) is an affine function of Y_t :

$$E[\exp(u \cdot Y_T) \mid \mathcal{F}_t] = \exp[\phi(\tau, u) + \psi(\tau, u) \cdot Y_t], \quad (3.7)$$

for all $(\tau, u) \in \mathbb{R}_+ \times i\mathbb{R}^d$, where $\tau := T - t$. The affine property will be used in this form throughout.

The admissible parameters associated with an affine process Y determine its generator and its functional characteristics F and R . The functional characteristics completely determine a regular affine process, since the functions ϕ and ψ satisfy generalized Riccati equations of the form $\partial_t \phi(t, u) = F(\psi(t, u))$ and $\partial_t \psi(t, u) = R(\psi(t, u))$; we refer the interested reader to the Appendix for further details.

Although the special form of the log-characteristic function of an affine process perfectly lends itself to tractable computations, we need to consider a class of processes for which formulae (3.6) or (3.7) extend to a broader subspace of \mathbb{C}^d than $i\mathbb{R}^d$.⁷ It is shown in Keller-Ressel [49, Chapter 3] that the functions ϕ and ψ characterizing the process Y have unique extensions to analytic functions on the interior $\text{int } \mathcal{E}_{\mathbb{C}}$ of the tube domain $\mathcal{E}_{\mathbb{C}} := \{(t, u) \in \mathbb{R}_+ \times \mathbb{C}^d : (t, \text{Re}(u)) \in \mathcal{E}\}$, where $\mathcal{E} := \{(t, v) \in \mathbb{R}_+ \times \mathbb{R}^d : v \in \mathcal{D}_{t+}\}$ and the set \mathcal{D}_{t+} is defined by $\mathcal{D}_{t+} := \bigcup_{s>t} \{z \in \mathbb{R}^d : \sup_{0 \leq r \leq s} E^y[\exp(z \cdot Y_r)] < \infty, \text{ for all } y \in D\}$. The extensions still satisfy the aforementioned Riccati equations and (3.6) and (3.7) extend to $\mathcal{E}_{\mathbb{C}}$.⁸ Recently, an alternative characterization of the extensibility of the affine transform formula (3.7) has been given in Keller-Ressel and Mayerhofer [50].

Equilibrium pricing formulae

We are now ready to state the main result of this section, that is a semi-explicit formula for the equilibrium price processes in an affine framework. For simplicity, we restrict the analysis to processes $Y = (V, X)$ with state space $D = \mathbb{R}_+ \times \mathbb{R}$, and we assume that the agents can trade K securities S^1, \dots, S^K with terminal payoffs

$$S_T^k = f^k(X_T), \quad (3.8)$$

for payoff functions $f^k : \mathbb{R} \rightarrow \mathbb{R}$. Under suitable integrability conditions our results carry over to more general payoff functions of the form $f^k(Y_T)$ and to affine processes on multi-dimensional state spaces. However, the resulting pricing formulae would be quite cumbersome and the Riccati equations that determine the processes' functional characteristics would no longer be solvable in closed form (the semi-explicit structure of the solution would be preserved, though). We define $f(x) := (f^1(x), \dots, f^K(x))$.

⁷By extension it is meant that the functions ϕ and ψ can be uniquely analytically extended to a suitable subspace of $\mathbb{R}_+ \times \mathbb{C}^d$.

⁸More precisely, Keller-Ressel [49, Lemma 3.12] states that this holds on the set $\{(t, u) \in \mathcal{E}_{\mathbb{C}} : |E^0[\exp(u \cdot Y_s)]| \neq 0, \text{ for all } s \in [0, t)\}$, whereas Keller-Ressel [49, Lemma 3.19] then yields that both sets coincide.

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Theorem 3.3. *Let $Y = (V, X)$ be an affine process on $\mathbb{R}_+ \times \mathbb{R}$, and suppose that the terminal payoffs of the securities are of the form (3.8). Suppose furthermore that there exists a vector of damping parameters $(\alpha^1, \dots, \alpha^K, \beta) \in \mathbb{R}^{K+1}$ such that the functions*

$$g_\zeta^k(x) := \exp(\alpha^k x) f^k(x) \exp(-\zeta \cdot f(x)) , \quad (3.9)$$

$$h_\zeta(x) := \exp(\beta x) \exp(-\zeta \cdot f(x)) , \quad (3.10)$$

and their respective Fourier transforms,

$$\hat{g}_\zeta^k(s) = \int_{\mathbb{R}} e^{-isy} g_\zeta^k(y) dy \quad \text{and} \quad \hat{h}_\zeta(s) = \int_{\mathbb{R}} e^{-isy} h_\zeta(y) dy ,$$

are integrable for all ζ in some neighborhood of $\tilde{\gamma}$, and that

$$(T, (0, -\alpha^k)) \in \mathcal{E} , \quad \text{for all } k, \quad \text{and} \quad (T, (0, -\beta)) \in \mathcal{E}. \quad (3.11)$$

Then, with $\hat{g}^k(s) \equiv \hat{g}_{\tilde{\gamma}}^k(s)$ and $\hat{h}(s) \equiv \hat{h}_{\tilde{\gamma}}(s)$, the following holds:

- (i) *The equilibrium price of S at time t is a function of $\tau := T - t$ and the current state of the process Y , and the price of the k -th security at time $t \in [0, T]$ is given by*

$$S_t^k = \frac{\int_{\mathbb{R}} \exp[\phi(\tau, (0, -\alpha^k + is)) + \psi(\tau, (0, -\alpha^k + is)) \cdot Y_t] \hat{g}^k(s) ds}{\int_{\mathbb{R}} \exp[\phi(\tau, (0, -\beta + is)) + \psi(\tau, (0, -\beta + is)) \cdot Y_t] \hat{h}(s) ds} . \quad (3.12)$$

Here, ϕ and ψ denote the analytic extensions of the functions introduced in Definition 3.2.

- (ii) *The equilibrium price process of S at time t can alternatively be computed by*

$$S_t^k = -\frac{\partial}{\partial \zeta^k} H(\zeta) / H(\tilde{\gamma}) \Big|_{\zeta=\tilde{\gamma}} . \quad (3.13)$$

Here, the function $H : \mathbb{R}^K \rightarrow \mathbb{R}$ is given by

$$H(\zeta) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp[\phi(\tau, (0, -\beta + is)) + \psi(\tau, (0, -\beta + is)) \cdot Y_t] \hat{h}_\zeta(s) ds .$$

Proof. Part 1: Pricing Formula (3.12). From Section 3.3.1 it is known that $Y = (V, X)$ satisfies

$$E[\exp(u \cdot Y_T) \mid \mathcal{F}_t] = \exp[\phi(\tau, u) + \psi(\tau, u) \cdot Y_t] , \quad (3.14)$$

for all $u = (u_v, u_x) \in \mathbb{C}^2$ such that $(T, u) \in \mathcal{E}_{\mathbb{C}}$, since the latter implies that (3.6) and thus (3.7) hold for all $t \in [0, T]$.

Let us first verify that (3.2) holds. Observe to this end that $E[\exp(-\tilde{\gamma} \cdot S_T)] = E[\exp(-\tilde{\gamma} \cdot f(X_T))] = E[\exp(-\beta X_T) h(X_T)]$. The Fourier transform \hat{h} defined in (3.10)

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exists and is integrable by assumption, allowing us to apply the Fourier inversion formula⁹ to obtain

$$h(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{isx} \hat{h}(s) ds,$$

dx -almost surely. With this at hand, we derive

$$\begin{aligned} 0 < E[\exp(-\tilde{\gamma} \cdot f(X_T))] &= \frac{1}{2\pi} E \left[\int_{\mathbb{R}} \exp[(-\beta + is)X_T] \hat{h}(s) ds \right] \\ &\leq E \left[\exp(-\beta X_T) \int_{\mathbb{R}} |\hat{h}(s)| ds \right] < \infty, \end{aligned} \quad (3.15)$$

since we required Y^T to be conservative, $(T, (0, -\beta)) \in \mathcal{E} \subseteq \mathcal{E}_{\mathbb{C}}$ and \hat{h} is integrable. Hence we have $\exp(-\tilde{\gamma} \cdot S_T) \in L^1(P)$ and the equilibrium pricing measure Q introduced in (3.3) is well-defined. Observe that $S_T \in L^1(Q)^K$ is proved analogously using $(T, (0, -\alpha^k)) \in \mathcal{E} \subseteq \mathcal{E}_{\mathbb{C}}$ and the integrability of all \hat{g}^k . Indeed, we obtain $E_Q[S_T^k] = E[f^k(X_T) \exp(-\tilde{\gamma} \cdot f(X_T))](E[\exp(-\tilde{\gamma} \cdot f(X_T))])^{-1}$ and

$$E[|f^k(X_T) \exp(-\tilde{\gamma} \cdot f(X_T))|] \leq E \left[\exp(-\alpha^k X_T) \int_{\mathbb{R}} |\hat{g}^k(s)| ds \right] < \infty. \quad (3.16)$$

Thus, following (3.4) and applying Bayes formula to (3.3), we obtain

$$S_t^k = E_Q[S_T^k | \mathcal{F}_t] = \frac{E[f^k(X_T) \exp(-\tilde{\gamma} \cdot f(X_T)) | \mathcal{F}_t]}{E[\exp(-\tilde{\gamma} \cdot f(X_T)) | \mathcal{F}_t]} \quad (3.17)$$

for the equilibrium price of the k -th security. Let us stipulate that an analogue to (3.15) holds for the conditional expectation $E[\cdot | \mathcal{F}_t]$ and thus we may apply Fubini's Theorem to exchange the order of integration and identify the denominator in (3.17) as

$$\begin{aligned} E[\exp(-\tilde{\gamma} \cdot f(X_T)) | \mathcal{F}_t] &= \frac{1}{2\pi} E \left[\int_{\mathbb{R}} \exp[(-\beta + is)X_T] \hat{h}(s) ds | \mathcal{F}_t \right] \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} E[\exp[(-\beta + is)X_T] | \mathcal{F}_t] \hat{h}(s) ds \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp[\phi(\tau, (0, -\beta + is)) + \psi(\tau, (0, -\beta + is)) \cdot Y_t] \hat{h}(s) ds. \end{aligned}$$

Notice that the affine transformation formula (3.14) holds, since $(T, (0, -\beta)) \in \mathcal{E}$. Applying the same arguments to the numerator in (3.17) in combination with the obser-

⁹See Dudley [26, Theorem 9.5.4].

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vation $E[f^k(X_T) \exp(-\tilde{\gamma} \cdot f(X_T))] = \frac{1}{2\pi} E[\int_{\mathbb{R}} \exp((- \alpha^k + is)X_T) \hat{g}^k(s) ds | \mathcal{F}_t]$ then yields the desired form of S_t^k in (3.12).

Part 2: Pricing Formula (3.13). We outline the details for $K = 1$, the rest follows by repeating the arguments for the partial derivative with respect to each ζ^k . So we assume we only have one security S with corresponding $\tilde{\gamma} \in \mathbb{R}$ affecting the density of the pricing measure Q . It follows that

$$\frac{dQ}{dP} = \frac{\exp(-\tilde{\gamma} S_T)}{E[\exp(-\tilde{\gamma} S_T)]} = \frac{\exp(-\tilde{\gamma} f(X_T))}{E[\exp(-\tilde{\gamma} f(X_T))]}$$

and the equilibrium price of S at time t can be obtained again by computing

$$S_t = \frac{E[f(X_T) \exp(-\tilde{\gamma} f(X_T)) | \mathcal{F}_t]}{E[\exp(-\tilde{\gamma} f(X_T)) | \mathcal{F}_t]}. \quad (3.18)$$

Recall from Part 1 that

$$\exp(-\tilde{\gamma} f(X_T)) \in L^1(P) \quad \text{and} \quad f(X_T) \exp(-\tilde{\gamma} f(X_T)) \in L^1(P), \quad (3.19)$$

due to the assumption of $(T, (0, -\alpha))$ and $(T, (0, -\beta))$ lying in \mathcal{E} . Since the set $\mathcal{E}_{\mathbb{C}}$ is open, compare Keller-Ressel [49, Lemmata 3.12 and 3.19], and due to the integrability assumptions on the functions \hat{h}_{ζ} and \hat{g}_{ζ} , the integrabilities in (3.19) even hold in some neighborhood $(\tilde{\gamma} - \varepsilon, \tilde{\gamma} + \varepsilon)$ of $\tilde{\gamma}$ where $\varepsilon > 0$. In particular, for any $0 < \tilde{\varepsilon} < \varepsilon$ holds

$$\begin{aligned} & \sup_{\zeta \in (\tilde{\gamma} - \tilde{\varepsilon}, \tilde{\gamma} + \tilde{\varepsilon})} |f(X_T) \exp(-\zeta f(X_T))| \\ & \leq |f(X_T) \exp(-[\tilde{\gamma} - \tilde{\varepsilon}] f(X_T))| \vee |f(X_T) \exp(-[\tilde{\gamma} + \tilde{\varepsilon}] f(X_T))| \end{aligned}$$

where the right-hand side is integrable by an analogue of (3.16). This allows us to differentiate the function $\zeta \mapsto E[\exp(-\zeta f(X_T)) | \mathcal{F}_t]$ at $\zeta = \tilde{\gamma}$ and we obtain

$$E[f(X_T) \exp(-\tilde{\gamma} f(X_T)) | \mathcal{F}_t] = - \frac{\partial}{\partial \zeta} E[\exp(-\zeta f(X_T)) | \mathcal{F}_t] \Big|_{\zeta = \tilde{\gamma}}, \quad (3.20)$$

as an application of Billingsley [6, Theorem 16.8] combined with the dominated convergence theorem for conditional expectations. On the other hand we know from Part 1 that the denominator in (3.18) can be computed by

$$\begin{aligned} & E[\exp(-\tilde{\gamma} f(X_T)) | \mathcal{F}_t] \\ & = \frac{1}{2\pi} \int_{\mathbb{R}} \exp[\phi(\tau, (0, -\beta + is)) + \psi(\tau, (0, -\beta + is)) \cdot Y_t] \hat{h}_{\tilde{\gamma}}(s) ds, \quad (3.21) \end{aligned}$$

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where we need the dependence of $\hat{h}(s) = \hat{h}_{\tilde{\gamma}}(s)$ on $\tilde{\gamma}$. Combining (3.20) and (3.21) yields

$$E[f(X_T) \exp(-\tilde{\gamma} f(X_T)) \mid \mathcal{F}_t] = -\frac{\partial}{\partial \zeta} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \exp[\phi(\tau, (0, -\beta + is))] \right. \\ \left. + \psi(\tau, (0, -\beta + is)) \cdot Y_t \right] \hat{h}_{\zeta}(s) ds \Big|_{\zeta=\tilde{\gamma}}. \quad \square$$

The benchmark case where only one security is in non-zero endowment-adjusted supply and its payoff function is linear, and all other securities are in zero endowment-adjusted supply, does not require Fourier transform methods, as shown by the following corollary.

Corollary 3.4. *Let the process Y and the functions $\hat{g}^k(s)$ be as in Theorem 3.3. Let us further assume that there is only one security, denoted by S^1 , in non-zero endowment-adjusted supply, that is*

$$\tilde{\gamma} = (\gamma(\eta^1 + n^1), 0, \dots, 0).$$

If furthermore $S_T^1 = X_T$ and $\tilde{\gamma}$ satisfies $(T, (0, -\tilde{\gamma}^1)) \in \mathcal{E}$, then the equilibrium price process of S^1 is given by

$$S_t^1 = [\partial_{u_x} \phi(\tau, u) + \partial_{u_x} \psi(\tau, u) \cdot Y_t] \Big|_{u=(0, -\tilde{\gamma}^1)}, \quad t \in [0, T], \quad (3.22)$$

where $\tau := T - t$ and ∂_{u_x} denotes the partial derivative with respect to the second argument of the vector $u = (u_v, u_x)$. Furthermore, whenever the remaining securities (S^2, \dots, S^K) satisfy the assumptions of Theorem 3.3, their price processes equal

$$S_t^k = \frac{1}{2\pi} \int_{\mathbb{R}} \exp[\Delta_{\tau}^{\alpha^k, \tilde{\gamma}^1}(\phi) + \Delta_{\tau}^{\alpha^k, \tilde{\gamma}^1}(\psi) \cdot Y_t] \hat{g}^k(s) ds, \quad (3.23)$$

for $k = 2, \dots, K$, and each $t \in [0, T]$. The shift operator $\Delta_t^{w,z}(\varphi)$ in (3.23) is defined by

$$\Delta_t^{w,z}(\varphi) := \varphi(t, (0, -w + is)) - \varphi(t, (0, -z)).$$

Proof. Expression (3.22) is an immediate consequence of (3.13) in Theorem 3.3 with $f(x) = x$, and the fact that there is no need of Fourier methods to compute the denominator $H(\tilde{\gamma})$ in the analogue to (3.18)

$$S_t^1 = \frac{E[X_T \exp(-\tilde{\gamma}^1 X_T) \mid \mathcal{F}_t]}{E[\exp(-\tilde{\gamma}^1 X_T) \mid \mathcal{F}_t]}, \quad (3.24)$$

since the affine transformation formula directly applies to the denominator in (3.24). We recall that $(T, (0, -\tilde{\gamma}^1)) \in \mathcal{E}$. Now we only need to compute $\frac{\partial}{\partial \zeta} E[e^{-\zeta X_T} \mid \mathcal{F}_t]$, the

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actual derivative in formula (3.13). However, from (3.7) it follows that

$$-\frac{\partial}{\partial \zeta} E[\exp(-\zeta X_T) \mid \mathcal{F}_t] \\ = \exp[\phi(\tau, u) + \psi(\tau, u) \cdot Y_t] [\partial_{u_x} \phi(\tau, u) + \partial_{u_x} \psi(\tau, u) \cdot Y_t] \Big|_{u=(0, -\zeta)} .$$

Combining the above with (3.24) yields

$$S_t^1 = [\partial_{u_x} \phi(\tau, u) + \partial_{u_x} \psi(\tau, u) \cdot Y_t] \Big|_{u=(0, -\tilde{\gamma}^1)} .$$

As to the remaining securities S^2, \dots, S^K , their price processes given in (3.23) directly follow from formula (3.12) in Theorem 3.3 and the discussion above. \square

3.3.2. Pricing of Call Options

We are now going to establish semi-explicit pricing formulae for European call options. The main challenge will be to find suitable “damping” parameters such that the Fourier methods of Theorem 3.3 can be applied. Specifically, we consider a market model with a single stock with terminal payoff $S_T = X_T$ and N call options on the stock with payoffs $C_T^i = (S_T - K_i)^+$, for $i = 1, \dots, N$, and strike prices $K_1 < \dots < K_N$. The stock and the options are traded simultaneously and hence collectively influence the equilibrium pricing kernel. The flattening parameters for S and C^k are denoted α and α^k , respectively; the corresponding weighted payoff functions are denoted g and g^k , respectively. We first state the pricing formula for the most general case of multiple simultaneously traded options in non-zero endowment-adjusted supply. The formulae are a direct application of Theorem 3.3. Subsequently, we consider the cases where either a single option in non-zero endowment-adjusted supply is traded, or multiple options in zero endowment-adjusted supply are traded.

Multiple, simultaneously traded options

Let us first consider the general case where $N > 0$ call options and one stock in non-zero endowment-adjusted supply are traded. As an illustration, we assume that throughout Sections 3.3.2 and 3.3.2 all supply-adjusted risk aversion parameters satisfy

$$\tilde{\gamma}^1 = \dots = \tilde{\gamma}^{N+1} = \gamma .$$

The pricing measure is then given by

$$\frac{dQ}{dP} = \frac{\exp\left(-\gamma \left(S_T + \sum_{i=1}^N (S_T - K_i)^+\right)\right)}{E\left[\exp\left(-\gamma \left(S_T + \sum_{i=1}^N (S_T - K_i)^+\right)\right)\right]} , \quad (3.25)$$

and the following result is an immediate consequence of Theorem 3.3.

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Theorem 3.5. *Given that α and β satisfy $\gamma < \alpha, \beta < (N+1)\gamma$ and (3.11), the equilibrium price of the underlying security S at time $t \in [0, T]$ is given by*

$$S_t = \frac{\int_{\mathbb{R}} \exp[\phi(\tau, (0, -\alpha + is)) + \psi(\tau, (0, -\alpha + is)) \cdot Y_t] \hat{g}(s) ds}{\int_{\mathbb{R}} \exp[\phi(\tau, (0, -\beta + is)) + \psi(\tau, (0, -\beta + is)) \cdot Y_t] \hat{h}(s) ds},$$

and the price of the k -th call option is given by

$$C_t^k = \frac{\int_{\mathbb{R}} \exp[\phi(\tau, (0, is)) + \psi(\tau, (0, is)) \cdot Y_t] \hat{g}^k(s) ds}{\int_{\mathbb{R}} \exp[\phi(\tau, (0, -\beta + is)) + \psi(\tau, (0, -\beta + is)) \cdot Y_t] \hat{h}(s) ds},$$

for $k = 1, \dots, N$. Here the functions \hat{g} , \hat{g}^k and \hat{h} are given by

$$\begin{aligned} \hat{g}(s) = & \sum_{j=1}^N \exp\left(\gamma \sum_{k=1}^{j-1} K_k\right) \exp[(-is + \alpha - j\gamma)K_j] \\ & \times \left[\left(\frac{-K_j\gamma}{(-is + \alpha - j\gamma)(-is + \alpha - (j+1)\gamma)} \right) \right. \\ & \left. + \left(\frac{1}{(-is + \alpha - (j+1)\gamma)^2} - \frac{1}{(-is + \alpha - j\gamma)^2} \right) \right] \end{aligned}$$

$$\begin{aligned} \hat{h}(s) = & \sum_{j=1}^N \exp\left(\gamma \sum_{k=1}^{j-1} K_k\right) \exp[(-is + \beta - j\gamma)K_j] \\ & \times \left[\frac{-\gamma}{(-is + \beta - j\gamma)(-is + \beta - (j+1)\gamma)} \right] \end{aligned}$$

$$\begin{aligned} \hat{g}^k(s) = & \exp\left(\gamma \sum_{h=1}^{k-1} K_h\right) \exp[(-is - k\gamma)K_k] \left[\frac{1}{(-is - (k+1)\gamma)^2} \right] \\ & + \sum_{j=k+1}^N \exp\left(\gamma \sum_{h=1}^{j-1} K_h\right) \exp[(-is - j\gamma)K_j] \left[\left(\frac{-(K_j - K_k)\gamma}{(-is - j\gamma)(-is - (j+1)\gamma)} \right) \right. \\ & \left. + \left(\frac{1}{(-is - (j+1)\gamma)^2} - \frac{1}{(-is - j\gamma)^2} \right) \right]. \end{aligned}$$

Proof. An application of Theorem 3.3 with $\alpha^k = 0$, for all $k = 1, \dots, N$, in addition to the observation that the Fourier transforms are all integrable functions yields the desired result. As to the second claim of integrability, straightforward calculations show that there exist constants $\hat{M}, \hat{z} > 0$, just depending on the model parameters, which give

$$\max_{f \in \{\hat{g}, \hat{h}, (\hat{g}^k)_{k=1}^N\}} \int_{\mathbb{R}} |f(s)| ds < \hat{M} \int_{\mathbb{R}} \frac{1}{s^2 + \hat{z}} ds < \infty. \quad \square$$

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The assumption $\gamma < \alpha, \beta < (N + 1)\gamma$ imposed on the damping factors ensures that the functions g and h of (3.9) and (3.10) allow for an integrable Fourier transform. In what follows, all model parameters have to be chosen such that (3.11) is satisfied and hence (3.7) applies. Further details are discussed below.

A single option model

The pricing kernel (3.25) and the Fourier transforms from Theorem 3.5 simplify considerably when only one option with strike $K > 0$ is traded. In this case the price processes (S_t) and (C_t) can be computed as in Theorem 3.5 by

$$\begin{aligned}\hat{g}(s) &= \exp[(\alpha - \gamma - is)K] \left[\frac{-K\gamma}{(-is - \gamma + \alpha)(-is - 2\gamma + \alpha)} \right. \\ &\quad \left. + \left(\frac{1}{(\alpha - 2\gamma - is)^2} - \frac{1}{(\alpha - \gamma - is)^2} \right) \right] \\ \hat{h}(s) &= \exp[(\beta - \gamma - is)K] \left(\frac{-\gamma}{(\beta - \gamma - is)(\beta - 2\gamma - is)} \right) \\ \hat{g}^1(s) &= \exp[-(is + \gamma)K] \frac{1}{(-is - 2\gamma)^2}.\end{aligned}$$

Options in zero endowment-adjusted supply

Let us finally consider the simplest situation in which all options are in zero endowment-adjusted supply. In this case, the equilibrium pricing kernel is independent of option payoffs and one only needs to find a suitable α corresponding to the weighted payoff function (3.9) in Theorem 3.3. The simple choice $\alpha = 0$ already guarantees that the Fourier-transform

$$\hat{g}^1(s) = \exp[-(is + \tilde{\gamma}^1)K] \frac{1}{(is + \tilde{\gamma}^1)^2}$$

of the function $g^1(x) := e^{-\tilde{\gamma}^1 x}(x - K)^+$ is integrable. The price process S is then given by (3.22), and the price of the call option at time $t \in [0, T]$ is given by

$$C_t = \frac{1}{2\pi} \int_{\mathbb{R}} \exp[\Delta_{\tau}^{0, \tilde{\gamma}^1}(\phi) + \Delta_{\tau}^{0, \tilde{\gamma}^1}(\psi_1)V_t + \Delta_{\tau}^{0, \tilde{\gamma}^1}(\psi_2)X_t] \hat{g}^1(s) ds,$$

with $\tau := T - t$ and $\Delta_{\tau}^{0, \tilde{\gamma}^1}$ defined in Corollary 3.4.

3.3.3. Equilibrium Dynamics in a Stochastic Volatility Model

By choosing the dynamics of Y according to the Heston stochastic volatility model Heston [36], it is possible to derive explicit equilibrium stock price formulae. Let $Y =$

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(V, X) be determined by

$$\begin{aligned} dV_t &= (\kappa - \lambda V_t)dt + \sigma\sqrt{V_t}dW_t^1 & V_0 &= v_0 \\ dX_t &= \mu dt + \sqrt{V_t}dW_t^2 & X_0 &= x_0, \end{aligned} \quad (3.26)$$

where (Ω, \mathcal{F}, P) is assumed to be rich enough to support the two-dimensional Brownian motion $W = (W^1, W^2)$.¹⁰ The market filtration is the augmentation of the filtration generated by Y . The parameters $\mu, \kappa, \lambda, \sigma > 0$ will be chosen appropriately later on. We initially assume that the agents are trading a single security S in unit endowment-adjusted supply with payoff $S_T = X_T$. We note that, unlike in the original model proposed by Heston, we do not model the log-payoff by (3.26). However, our approach is justified by considering only short time horizons. Since the above additive Heston model is affine and allows for explicit solutions of the functions ϕ and ψ , we apply the results obtained in Sections 3.2 and 3.3 to compute the equilibrium price S_t at time $t \in [0, T]$ in closed form as a function of Y_t .

Theorem 3.6. *Let $\theta(\gamma)$ be defined by*

$$\theta(\gamma) = \begin{cases} \sqrt{\lambda^2 - \sigma^2 \gamma^2} & \text{if } \gamma < \frac{\lambda}{\sigma} \\ i\sqrt{\sigma^2 \gamma^2 - \lambda^2} & \text{if } \gamma > \frac{\lambda}{\sigma} \end{cases}.$$

Suppose that γ is such that T satisfies

$$T < \begin{cases} +\infty & \gamma < \frac{\lambda}{\sigma} \\ \frac{2}{|\theta(\gamma)|} \left(\arctan \frac{|\theta(\gamma)|}{-\lambda} + \pi \right) & \gamma > \frac{\lambda}{\sigma} \end{cases}. \quad (3.27)$$

Then we have that, with $\tau := T - t$, $\theta := \theta(\gamma)$ and $\theta' := \frac{\partial}{\partial \gamma} \theta(\gamma)$, the equilibrium price process S is given by

$$S_t = T(\tau, \gamma) - \gamma \Gamma(\tau, \gamma) V_t + X_t, \quad (3.28)$$

for $t \in [0, T]$, and where

$$\begin{aligned} T(\tau, \gamma) &= \frac{2\kappa}{\sigma^2 \theta} \left[\theta(e^{\theta\tau} + 1) + \lambda(e^{\theta\tau} - 1) \right]^{-1} \left[\left(\theta(e^{\theta\tau} + 1) + \lambda(e^{\theta\tau} - 1) \right) \left(\theta' - \frac{1}{2}\sigma^2 \gamma \tau \right) \right. \\ &\quad \left. - \theta \left(\theta'(e^{\theta\tau} + 1) + \tau e^{\theta\tau} (\lambda \theta' - \gamma \sigma^2) \right) \right] \end{aligned}$$

¹⁰The more general case of correlated Brownian motions could be included in (3.26) by considering $W^3 := \rho W^1 + \sqrt{1 - \rho^2} W^2$ instead of W^2 . We choose zero correlation in order to keep the notation simple.

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$$\begin{aligned} \Gamma(\tau, \gamma) = & \left[\theta (e^{\theta\tau} + 1) + \lambda (e^{\theta\tau} - 1) \right]^{-1} \left[\left(2 (e^{\theta\tau} - 1) - \gamma\tau\theta'e^{\theta\tau} \right) \right. \\ & \left. + \gamma (e^{\theta\tau} - 1) \left(\theta' (e^{\theta\tau} + 1) + \tau e^{\theta\tau} (\lambda\theta' + \gamma\sigma^2) \right) \left(\theta (e^{\theta\tau} + 1) + \lambda (e^{\theta\tau} - 1) \right)^{-1} \right]. \end{aligned}$$

Proof. The process $Y = (V, X)$ belongs to a subclass of affine processes, namely to the \mathbb{R}^2 -valued affine diffusions. That is, Y is a solution to the stochastic differential equation $dY_t = \mu(Y_t)dt + \rho(Y_t)dW_t$, with $Y_0 = y_0$, for a continuous function $b : D \rightarrow \mathbb{R}^2$ and a measurable function $\rho : D \rightarrow \mathbb{R}^{2 \times 2}$ such that $y \mapsto \rho(y)\rho(y)^T$ is continuous.¹¹ In particular, the set $\text{int } \mathcal{D}_{0+}$ from Section 3.3.1 is non-empty and thus the affine transform formula can be extended. See for instance the discussion on explosion times of the Heston model in Friz and Keller-Ressel [32]. Furthermore, the process Y is conservative and, hence, so is the stopped process Y^T . Combining (A.3) in the appendix with the fact that the generator of (V, X) is determined by its diffusion matrix $\rho\rho^T$ and its drift vector b , we identify the admissible parameters in (A.1), (A.2) and (A.3), where the parts connected with jumps do not play a role here. Hence we conclude that the conditional characteristic function of Y allows a representation as follows

$$E [\exp (u \cdot Y_T) \mid \mathcal{F}_t] = \exp [\phi(\tau, u) + \psi(\tau, u) \cdot Y_t], \quad (3.29)$$

whenever $(T, u) = (T, (u_v, u_x)) \in \mathcal{E}_{\mathbb{C}}$, so in particular for $(T, (u_v, u_x)) \in \mathcal{E}$. The functions ϕ and ψ satisfy the following system of Riccati equations

$$\begin{aligned} \partial_t \phi(t, u) &= \kappa \psi_1(t, u) + \mu \psi_2(t, u) & \phi(0, u) &= 0 \\ \partial_t \psi_1(t, u) &= \frac{1}{2} \sigma^2 \psi_1(t, u)^2 - \lambda \psi_1(t, u) + \frac{1}{2} \psi_2(t, u)^2 & \psi_1(0, u) &= u_v \\ \partial_t \psi_2(t, u) &= 0 & \psi_2(0, u) &= u_x. \end{aligned} \quad (3.30)$$

A solution to the above system (3.30), evaluated at the vector $u = (0, u_x)$, is given by¹²

$$\begin{aligned} \phi(t, (0, u_x)) &= \frac{2\kappa}{\sigma^2} \log \left(\frac{2\theta(u_x) \exp \left(\frac{\theta(u_x) + \lambda}{2} t \right)}{\theta(u_x)(e^{\theta(u_x)t} + 1) + \lambda(e^{\theta(u_x)t} - 1)} \right) + \mu u_x t \\ \psi_1(t, (0, u_x)) &= \frac{u_x^2 (e^{\theta(u_x)t} - 1)}{\theta(u_x)(e^{\theta(u_x)t} + 1) + \lambda(e^{\theta(u_x)t} - 1)} \\ \psi_2(t, (0, u_x)) &= u_x \end{aligned}$$

¹¹We emphasize that we would not need the complete theory of general affine processes including various possible behavior of jumps, had we only considered pure diffusion processes, since it is shown in Filipovic [31, Theorem 10.1] that every diffusion Markov process with continuous diffusion matrix is affine if and only if the functions b and $\rho\rho^T$ are affine in the state variable and the solutions ϕ and ψ of the Riccati equations satisfy $\text{Re}(\phi(t, u) + \psi(t, u) \cdot y) \leq 0$, for all $y \in D$ and $(t, u) \in \mathbb{R}_+ \times i\mathbb{R}^d$. Our equilibrium approach can cover more sophisticated models than pure diffusions though.

¹²Compare Filipovic [31, Lemma 10.12]. For $u_x = \lambda/\sigma$ we set $\psi_1(t, (0, \frac{\lambda}{\sigma})) = t/(2 + \lambda t)$, resembling the limit and still satisfying $\psi_1(0, (0, \lambda/\sigma)) = 0$.

where

$$\theta(u_x) = \begin{cases} \sqrt{\lambda^2 - \sigma^2 u_x^2} & \text{if } |u_x| < \frac{\lambda}{\sigma} \\ i\sqrt{\sigma^2 u_x^2 - \lambda^2} & \text{if } |u_x| > \frac{\lambda}{\sigma} \end{cases}.$$

Following Friz and Keller-Ressel [32] and recalling that $\lambda > 0$, we distinguish two different cases

$$t^+(u_x) = \begin{cases} +\infty & |u_x| < \frac{\lambda}{\sigma} \\ \frac{2}{|\theta(u_x)|} \left(\arctan \frac{|\theta(u_x)|}{-\lambda} + \pi \right) & |u_x| > \frac{\lambda}{\sigma} \end{cases}$$

such that $(T, (0, u_x)) \in \mathcal{E} \subseteq \mathcal{E}_{\mathbb{C}}$, for all $T \leq t^+(u_x)$.¹³ Hence, as long as $T < t^+(u_x)$, formula (3.29) holds for all $u = (0, u_x)$, where $u_x \in \mathbb{R}$. It now follows from (3.22) in Corollary 3.4 that, for all $t \in [0, T]$,

$$S_t = [\partial_{u_x} \phi(\tau, u) + \partial_{u_x} \psi_1(\tau, u) V_t + \partial_{u_x} \psi_2(\tau, u) X_t] \Big|_{u=(0, -\gamma)}. \quad (3.31)$$

Next, we need to compute the derivatives of $\phi(t, u)$ and $\psi(t, u)$ with respect to u_x . Of course we have $\partial_{u_x} \psi_2(\tau, u) \equiv 1$ and a straightforward calculation yields, with $\theta := \theta(-\gamma)$ and $\theta' := [\partial_{u_x} \theta](-\gamma)$,

$$\partial_{u_x} \phi(\tau, (0, -\gamma)) = T(\tau, \gamma) \quad \text{and} \quad \partial_{u_x} \psi_1(\tau, (0, -\gamma)) = -\gamma \Gamma(\tau, \gamma).$$

This, together with (3.31), is (3.28), the proof is complete. \square

We note that (3.27) ensures that (3.11) in Theorem 3.3 is satisfied, which, in combination with the discussion in Section 3.3.2, allows us to study the impact of the model parameters in a framework comprising European-style options. In particular, we illustrate within the Heston framework the effect of the parameters γ and σ on implied volatilities using the formulae obtained in Theorem 3.5. To this end, we consider a setting with one underlying asset and fifteen simultaneously traded call options written on it, all affecting the pricing density. In Figure 3.1, four different implied volatility curves are shown, corresponding to four different values of the risk aversion γ . We see that, especially for in-the-money options, higher risk aversion yields a higher level of implied volatility. The more risk-averse the representative agent is, the more in-the-money options are appreciated as good hedges against possibly low values of the underlying. In the recent work Sircar and Sturm [62] the impact of market risk aversion on put option implied volatilities is investigated by means of indifference pricing by dynamic convex risk measures and asymptotic methods.

The implied volatility curves for two different choices of the vol-of-vol parameter σ in (3.26) are shown in Figure 3.2. We observe a significant increase in implied volatility when changing from the low value (blue curve) to the higher one (red curve). That is due to the fact that a high value of σ increases the probability of S_T taking on extreme

¹³Basically, this is exactly the time interval on which the solutions of the Riccati equations do not explode.

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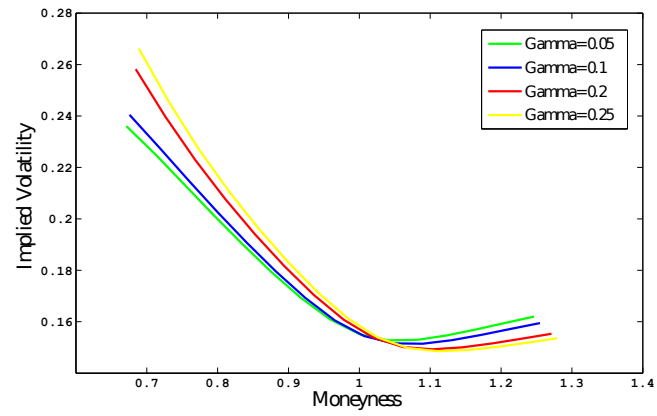


Figure 3.1.: Implied volatility curves with varying risk aversion γ

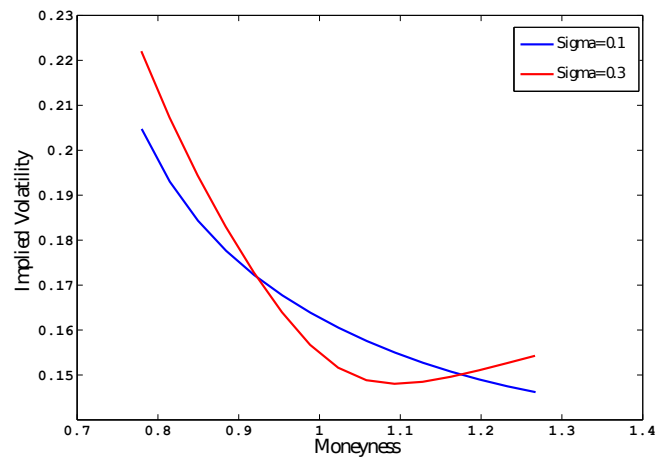


Figure 3.2.: Implied volatility curves with varying vol-of-vol σ

tail values and hence rendering even out-of-the-money options attractive instruments.¹⁴

3.3.4. Equilibrium Dynamics in a Pure Jump Ornstein-Uhlenbeck Setting

In order to include the presence of jumps into the discussion of equilibrium prices, we consider now a single stock with terminal payoff $S_T = X_T$ where X is an Ornstein-Uhlenbeck process with a pure jump component as Lévy part¹⁵:

$$dX_t = -\lambda(X_t - \mu)dt + dJ_t \quad , \quad X_0 = x_0 .$$

Here, J is an adapted compound Poisson process with intensity $\kappa > 0$ and jump distribution $\nu(dx) = \frac{1}{2}\theta \exp(-\theta|x|)dx$.¹⁶ The parameters μ and λ describe the long term mean and the mean reversion rate, respectively. In this one-dimensional setting the equations for the functional characteristics F and R are given by

$$F(u) = \lambda\mu u + \frac{\kappa u^2}{\theta^2 - u^2} \quad \text{and} \quad R(u) = -\lambda u , \quad (3.32)$$

see (A.1) and (A.2). Combining (3.32) with (A.4) and (A.5), we deduce that the functions ϕ and ψ satisfy the following system of Riccati equations

$$\begin{aligned} \partial_t \phi(t, u) &= \lambda\mu\psi(t, u) + \frac{\kappa\psi^2(t, u)}{\theta^2 - \psi^2(t, u)} , & \phi(0, u) &= 0 \\ \partial_t \psi(t, u) &= -\lambda\psi(t, u) , & \psi(0, u) &= u , \end{aligned}$$

which allows for the explicit solutions

$$\phi(t, u) = \frac{\kappa}{2\lambda} \log \left(\frac{\theta^2 - u^2 e^{-2\lambda t}}{\theta^2 - u^2} \right) + \mu u (1 - e^{-\lambda t}) \quad \text{and} \quad \psi(t, u) = u e^{-\lambda t} .$$

Thus, (3.7) holds, as long as $u \in \mathbb{R} \setminus \{-\theta, \theta\}$ and $T < t^*(u)$, with

$$t^*(u) = \begin{cases} +\infty & |u| < \theta \\ -\frac{1}{2\lambda} \log(\frac{\theta^2}{u^2}) & |u| > \theta \end{cases} . \quad (3.33)$$

This, together with Corollary 3.4, allows us to formulate the following:

Proposition 3.7. *If $|\tilde{\gamma}| \neq \theta$ and $T < t^*(-\tilde{\gamma})$, where t^* is as in (3.33), then, with*

¹⁴ For the Figures 3.1 and 3.2, the following parameters were used for the numerical computations: $\mu = 0.1$, $\kappa = 0.006$, $\lambda = 0.2$, $T = 0.5$, $t = 0$, $(x_0, v_0) = (1, 0.03)$. In Figure 3.1, we set $\sigma = 0.3$, whereas in Figure 3.2, $\gamma = 0.2$ was used.

¹⁵ This is a specific subclass of basic affine processes, compare Duffie and Singleton [28, Section A.2].

¹⁶ More precisely, $J_t = \sum_{i=0}^{N_t} b_i D_i$, where N_t is a Poisson process with intensity κ , D_i are exponentially distributed i.i.d. random variables with jumps of mean $\frac{1}{\theta} > 0$, and b_i are i.i.d. Bernoulli random variables with $P[b_1 = 1] = P[b_1 = -1] = 0.5$.

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$\tau := T - t$, the equilibrium price process S is given by

$$S_t = \left[\frac{\kappa \theta^2 \tilde{\gamma} (e^{-2\lambda\tau} - 1)}{\lambda(\theta^2 - \tilde{\gamma}^2)(\theta^2 - u^2 e^{-2\lambda\tau})} + \mu(1 - e^{-\lambda\tau}) \right] + e^{-\lambda\tau} X_t, \quad t \in [0, T].$$

In the following we illustrate the influence of the parameters γ , κ and θ on option implied volatilities. Figure 3.3 illustrates the dependence of implied volatilities on the jump parameters for fixed risk aversion. The red curve corresponds to smaller jumps

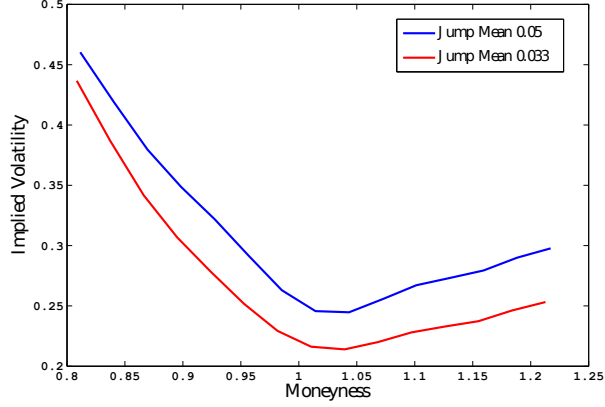


Figure 3.3.: Implied volatility curves with varying jump mean $1/\theta$ and intensity κ

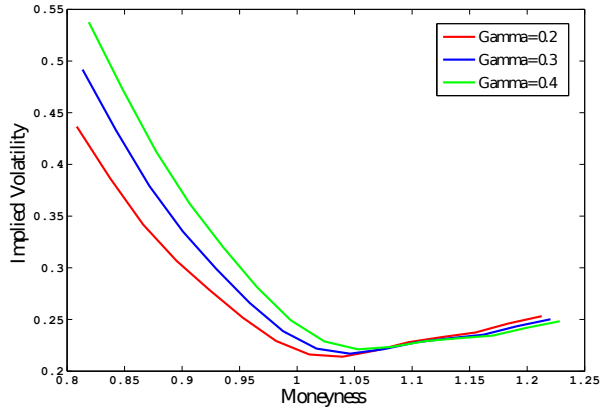


Figure 3.4.: Implied volatility curves with varying risk aversion γ

arriving at a high frequency $((\kappa, \frac{1}{\theta}) = (30, \frac{1}{30}))$, whereas the blue one was obtained considering higher jumps at a lower frequency $((\kappa, \frac{1}{\theta}) = (20, \frac{1}{20}))$. Increasing the mean jump height distinctly lifts the level of implied volatility, since the probability of S_T

taking extreme values is higher that way. We further note that an affine model including jumps seems in general more suitable to reproduce the right-hand side smile observed in real market data. In Figure 3.4 in turn, we observe that an increase in implied volatility for in-the-money call options is caused by increasing risk aversion, similar to the stochastic volatility model discussed before.¹⁷

3.4. Information-Based Equilibrium Pricing

In this section, we propose another method to model the market filtration based on the information-based asset pricing approach of Brody et al. [11] and Hoyle et al. [42]. This approach is based on the modeling of cash flows and the explicit construction of market filtrations, which can be naturally embedded in the equilibrium pricing model considered in the present chapter. The key idea is that, instead of assuming from the outset some abstract filtration representing the information available to the market, processes carrying market-relevant information are explicitly constructed, and a distinction between “genuine” information and market noise is made. The equilibrium dynamics is then computed by using the special form of the pricing measure obtained in Section 3.2, by assuming an a priori distribution of the market factor determining the terminal payoff, and by updating a posteriori distributions about the assets’ payoffs obtained by a version of Bayes formula.

3.4.1. Setup and Equilibrium Pricing Formula

We assume that the probability space (Ω, \mathcal{F}, P) supports a N -dimensional Brownian motion B together with N independent random market factors $(X_i)_{i=1}^N$, all independent of B , and define $S_T^k = f^k(X_1, \dots, X_N)$. The agents know the a priori distributions ν^i of all X_i . With each market factor X_i , we associate an observable process $(\xi_t)_{t \in [0, T]}$, the so-called information process. The information processes are defined by

$$\xi_t^i = \sigma_i X_i t + \beta_t^i, \quad t \in [0, T], \quad (3.34)$$

where the independent standard Brownian bridges β^i on $[0, T]$ are defined in terms of B as solutions to the SDEs

$$d\beta_t^i = -\frac{\beta_t^i}{T-t} dt + dB_t^i, \quad \beta_0^i = 0, \quad (3.35)$$

for $t \in [0, T)$, and $\beta_T^i = 0$. Looking at the different components of the processes (3.34), we identify the part $\sigma_i X_i t$ containing real information about the realization of a market factor revealed over time, and the bridge part representing market noise. The speed at which the outcome of X_i is revealed is governed by the information rate σ_i . The

¹⁷The remaining parameters in Figures 3.3 and 3.4 were chosen as $(\mu, \lambda, T, t, x_0) = (1, 2, 0.1, 0, 1)$. In Figure 3.3 we set $\gamma = 0.2$, whereas the jump parameters were chosen as $(\kappa, \frac{1}{\theta}) = (30, \frac{1}{30})$ in Figure 3.4. As before, we considered 15 simultaneously traded call options.

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information processes capture the flow of information available to the market agents, and thus generate the market filtration:

$$\mathcal{F}_t = \sigma(\xi_s^1, \dots, \xi_s^N, s \leq t), \quad t \in [0, T].$$

By construction, S_T is \mathcal{F}_T -measurable, and at each time $t \in [0, T]$, the equilibrium price S_t will be determined using the results of Section 3.2.

Theorem 3.8. *Assume that all a priori distributions ν^i allow for a density with respect to the Lebesgue-measure denoted by $v^i(x)$, respectively. If in addition the functions $(f^k)_{k=1}^K$ and the a priori densities $(v^i)_{i=1}^N$ are such that (3.2) is satisfied, then, for $t < T$, the equilibrium price process of the k -th security is given by*

$$S_t^k = \frac{\int_{\mathbb{R}^N} z(x_1, \dots, x_N) f^k(x_1, \dots, x_N) \pi_t^1(x_1) \cdots \pi_t^N(x_N) dx_1 \dots dx_N}{\int_{\mathbb{R}^N} z(x_1, \dots, x_N) \pi_t^1(x_1) \cdots \pi_t^N(x_N) dx_1 \dots dx_N}, \quad (3.36)$$

where the function z is defined by

$$z(\cdot) = \exp \left[- \sum_{l=1}^K \tilde{\gamma}^l f^l(\cdot) \right]. \quad (3.37)$$

The regular conditional density function π_t^i associated with the i -th market factor is given by

$$\pi_t^i(x) = \frac{v^i(x) \exp \left[\frac{T}{T-t} (\sigma_i x \xi_t^i - \frac{1}{2} (\sigma_i x)^2 t) \right]}{\int_{\mathbb{R}} v^i(y) \exp \left[\frac{T}{T-t} (\sigma_i y \xi_t^i - \frac{1}{2} (\sigma_i y)^2 t) \right] dy}. \quad (3.38)$$

Proof. By assumption, the conditions of Theorem 3.1 are satisfied. Recall that the equilibrium price is obtained by the change of measure from P to Q , that is:

$$\begin{aligned} S_t^k &= E_Q [S_T^k | \mathcal{F}_t] = E_Q [f^k(X_1, \dots, X_N) | \mathcal{F}_t] \\ &= E \left[\frac{dQ}{dP} f^k(X_1, \dots, X_N) | \mathcal{F}_t \right] E \left[\frac{dQ}{dP} | \mathcal{F}_t \right]^{-1}. \end{aligned}$$

By (3.3), we know that $\frac{dQ}{dP}$ is a function of S_T and hence of X_1, \dots, X_N , which is given in (3.37). Then we compute the regular conditional distribution of (X_1, \dots, X_N) given $(\xi_t^1, \dots, \xi_t^N)$. Using the independence of the market factors, the Markov property of ξ , the Bayes formula, and observing that, given $(X_1, \dots, X_N) = (x_1, \dots, x_N)$, ξ_t^i is Gaussian with mean $\sigma_i x_i t$ and variance $\frac{tT}{T-t}$, yields (3.38). \square

3.4.2. Innovation Processes and Equilibrium Market Price of Risk

Let us consider $K = N = 1$, and in particular the case $S_T = X$ with corresponding information process $\xi_t = \sigma X t + \beta_t$, for $t \in [0, T]$. We assume that the market factor X is such that the conditions of Theorem 3.8 are satisfied. Formula (3.36) now reduces to

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$$S_t = \frac{E[S_T \exp(-\tilde{\gamma} S_T) \mid \mathcal{F}_t]}{E[\exp(-\tilde{\gamma} S_T) \mid \mathcal{F}_t]} = \frac{\int x \exp(-\tilde{\gamma} x) \pi_t(x) dx}{\int \exp(-\tilde{\gamma} x) \pi_t(x) dx}. \quad (3.39)$$

Results from general filtering theory guarantee the existence of a P -Brownian motion W on $[0, T)$, adapted to the market filtration generated by ξ . The so-called innovations process W associated with the information process ξ satisfies

$$W_t = \xi_t - \int_0^t \left[\frac{1}{T-s} (\sigma T E[X \mid \mathcal{F}_s] - \xi_s) \right] ds, \quad t < T. \quad (3.40)$$

Thus, instead of having to assume the existence of Brownian motions as drivers for the prices, they rather emerge naturally from within the information-driven structure, as the following proposition shows.

Proposition 3.9. *Assume that $g(X)$ and $h(X)$ belong to $L^2(P)$ where g and h are defined by $g(x) = x \exp(-\tilde{\gamma} x)$ and $h(x) = \exp(-\tilde{\gamma} x)$, respectively. Then the equilibrium dynamics of $(S_t)_{t < T}$ are given by*

$$dS_t = \frac{\sigma T}{T-t} \text{Var}_t^Q(X) \left[\frac{\sigma T}{T-t} (E[X \mid \mathcal{F}_t] - S_t) dt + dW_t \right] \quad (3.41)$$

where

$$\text{Var}_t^Q(X) := E_Q[X^2 \mid \mathcal{F}_t] - (E_Q[X \mid \mathcal{F}_t])^2 \quad (3.42)$$

is the conditional variance of X under the measure Q defined in (3.3).

Proof. The integrability assumptions on X together with Liptser and Shiryaev [53, Theorem 7.17] yield that the innovation Brownian motion W_t in (3.40) is well-defined for $t < T$. By the Fujisaki-Kallianpur-Kunita Theorem, see Bain and Crisan [3, Proposition 2.31], both expressions appearing in (3.39) allow for a representation with respect to W . Furthermore, we even know the structure of the integrands. Specifically, for every function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\varphi(X) \in L^2(P)$ and for $t < T$, we obtain that

$$E[\varphi(X) \mid \mathcal{F}_t] = E[\varphi(X)] + \int_0^t \frac{\sigma T}{T-u} V_u^\varphi dW_u, \quad (3.43)$$

where V_t^φ , the conditional covariance of the market factor with the function φ , is given by

$$V_t^\varphi = E[\varphi(X)X \mid \mathcal{F}_t] - E[\varphi(X) \mid \mathcal{F}_t] E[X \mid \mathcal{F}_t], \quad (3.44)$$

as shown in Brody et al. [11, Section V]. The dynamics (3.41) then follow by (3.43) in combination with (3.44) and an application of the Itô product rule to (3.39). \square

The expressions $E[X \mid \mathcal{F}_t]$ and $\text{Var}_t^Q(X)$ can be worked out semi-explicitly by means of (3.42), the integral formula (3.39), and the regular conditional density $\pi(x)$ defined in (3.38). They are functions of the pair (t, ξ_t) and triplet $(t, \xi_t, \tilde{\gamma})$, respectively, due to

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(3.3) and the Markov property of the information process. By an application of Lévy's characterization of Brownian motion, it can be shown that the process $(W_t^Q)_{t < T}$ defined by

$$dW_t^Q = \frac{\sigma T}{T-t} (E[X | \mathcal{F}_t] - S_t) dt + dW_t$$

is an $((\mathcal{F}_t), Q)$ -Brownian motion. Thus, (3.41) confirms that $(S_t)_{t < T}$ is an $((\mathcal{F}_t), Q)$ -martingale.

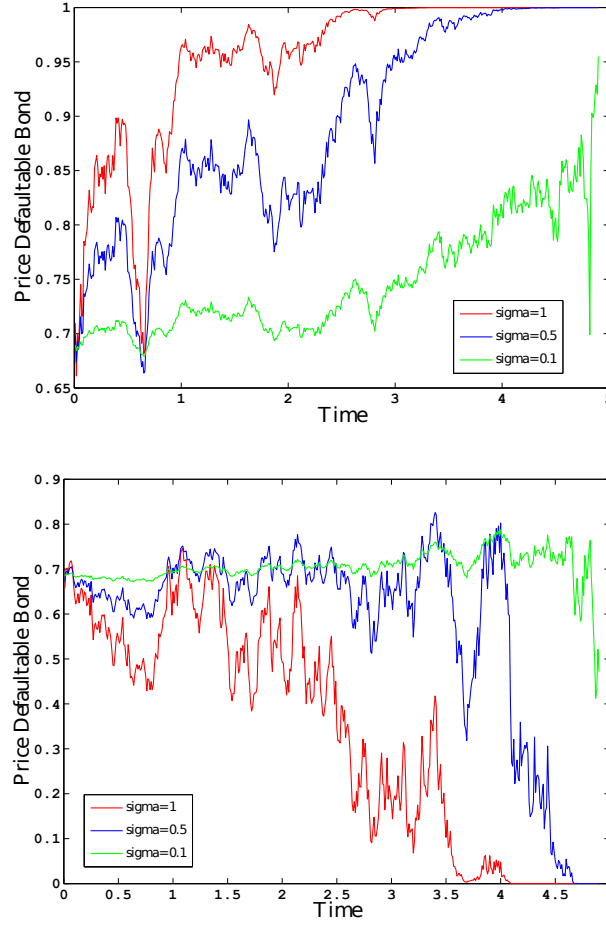
3.4.3. Pricing Credit-Risky Securities

In this section, we illustrate the impact of the “noisyness” of information and of the market risk aversion on the equilibrium prices of a credit-sensitive security within a simple benchmark model, see Brody et al. [10], where the a-priori distribution of $S_T = X$ is discrete: $S_T \in \{x_0, x_1\} = \{0, 1\}$. We denote by $p_0 := P[X = 0]$ the probability of default. Due to the discrete payoff structure, formula (3.36) simplifies and allows us to examine the impact of model parameters, such as the information flow rate or the risk aversion, on the equilibrium price of S . The price of the security threatened by default can be obtained in closed form analogously to (3.36) and is given by

$$S_t = \frac{p_1 x_1 \exp(-\tilde{\gamma} x_1) \exp\left[\frac{T}{T-t} (\sigma x_1 \xi_t - \frac{1}{2}(\sigma x_1)^2 t)\right]}{\sum_{i=0,1} p_i \exp(-\tilde{\gamma} x_i) \exp\left[\frac{T}{T-t} (\sigma x_i \xi_t - \frac{1}{2}(\sigma x_i)^2 t)\right]}, \quad t < T.$$

Figure 3.5 shows the impact of σ on the price of a defaultable bond, where the probability of default is chosen to be $p_0 = 0.2$. In the upper graphic the bond does not default, whereas in the lower graphic we considered the situation of a default. In both cases, a low information flow rate (green curve, $\sigma = 0.1$) leads to a rather late adjustment of the equilibrium price process towards the prevailing terminal value, while the red curve ($\sigma = 1$) reacts earlier to the information about the outcome of X . The influence of the risk aversion $\tilde{\gamma}$ on defaultable bond prices is demonstrated in Figure 3.6. It is evident that a higher risk aversion leads to a more careful evaluation of the bond, since the possibility of a default is taken more into account. This effect occurs in both depicted scenarios, where in the upper and lower figure the information rate σ is chosen to be $\sigma = 0.2$ and $\sigma = 0.5$, respectively. Note however that for the case of a low information rate (upper graphic) the initial price difference turns out to be smaller, because both agents, the more and less risk-averse one, consider the information to be noisier, hence less valuable, and thus give the bond a lower price.¹⁸

¹⁸The following parameters were used for the simulations shown in Figures 3.5 and 3.6: $P[X = x_1] = 0.8$, $T = 5$. The price process is shown for $t \in [0, 4.9]$. In Figure 3.5 we set $\tilde{\gamma} = 0.6$.


 Figure 3.5.: Defaultable bond prices: influence information rate σ

3.4.4. One-Dimensional, Exponentially-Distributed Terminal Cash Flow

We illustrate how, for particular choices of v and f , the formulae (3.38) and (3.39) can be worked out explicitly. We assume $f(x) = x$, corresponding to the assets payoff itself being the market factor. Furthermore, the a priori distribution of S_T , the cash flow at time T , is assumed to be exponential.

Corollary 3.10. *Assume that the a-priori distribution of $S_T = X$ is of the exponential form, that is, $v(x) = (1_{\{x \geq 0\}}/\kappa) \exp(-x/\kappa)$ for some $\kappa > 0$. If $\tilde{\gamma} > \kappa - 1$, then the*

3. Continuous Equilibrium in Affine and Information-Based CAPMs

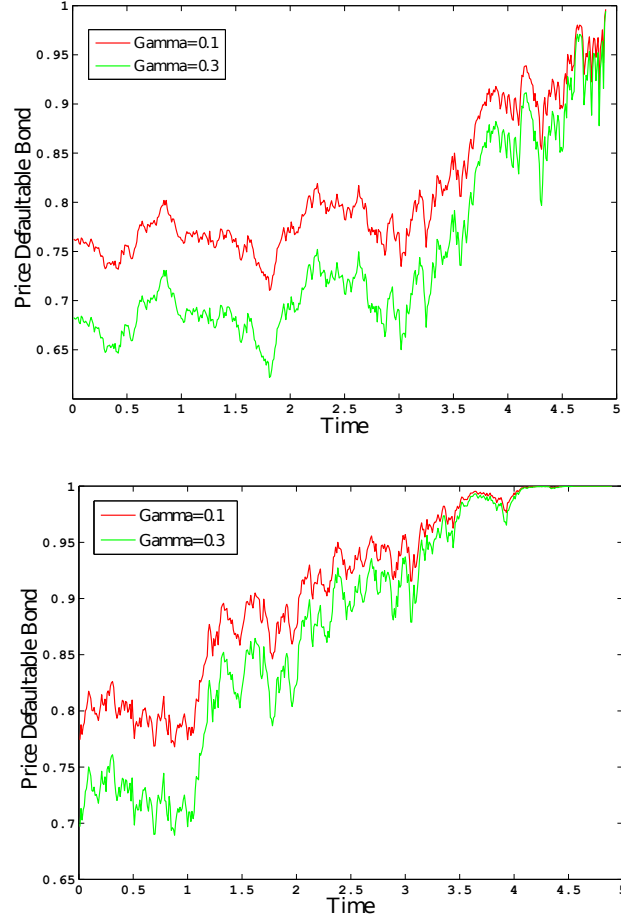


Figure 3.6.: Defaultable bond prices: influence risk aversion γ

equilibrium price at time $t < T$ is given by

$$S_t = \left[\frac{\exp\left(-\frac{1}{2}B_t^2/A_t\right)}{\sqrt{2\pi A_t} \mathcal{N}(B_t/A_t)} + \frac{B_t}{A_t} \right], \quad (3.45)$$

where

$$A_t = \sigma^2 t T / (T - t) \quad , \quad B_t = \sigma T \xi_t / (T - t) - \frac{\tilde{\gamma} \kappa + 1}{\kappa}, \quad (3.46)$$

and $\mathcal{N}(x)$ denotes the standard normal distribution function.

Proof. The relation $\tilde{\gamma} > \kappa - 1$ ensures that the assumptions of Theorem 3.1 are met. It

remains to apply Theorem 3.8 and explicitly work out the integrals in

$$\frac{\int_0^\infty x (1/\kappa) \exp(-x/\kappa) \exp(-\tilde{\gamma}x) \exp\left[\frac{T}{T-t} (\sigma x \xi_t - \frac{1}{2}(\sigma x)^2 t)\right] dx}{\int_0^\infty (1/\kappa) \exp(-x/\kappa) \exp(-\tilde{\gamma}x) \exp\left[\frac{T}{T-t} (\sigma x \xi_t - \frac{1}{2}(\sigma x)^2 t)\right] dx},$$

which is done by combining Brody et al. [11, Section VII] and (3.47) below, resulting in formulae (3.45) and (3.46). \square

Since the pricing measure depends only on the terminal cash-flow as a consequence of the attainable endowments, changing from P to Q could be interpreted as a different view \tilde{v} of the representative agent on the a-priori-distribution of S_T . More precisely, under Q the cash-flow S_T is exponentially distributed with new parameter $(\tilde{\gamma}\kappa + 1)/\kappa$, also appearing in (3.46), which can be seen by working out the adjusted density

$$\tilde{v}(x) = \frac{\exp(-\tilde{\gamma}x)v(x)}{\int \exp(-\tilde{\gamma}y)v(y)dy}. \quad (3.47)$$

Addendum to Section 3.3: Regular Affine Processes

This proposition concerning the characterization of a regular affine process by its admissible parameters is stated without proof and we refer to Duffie et al. [29, Theorem 2.7] or Keller-Ressel [49, Theorem 2.6 and Equations (2.2a),(2.2b)] for two different approaches to prove it.

Proposition A.1. *Let Y be a regular affine process with state space D . Let F and R be as in Definition 3.2. Then there exists a set of admissible parameters $(A, A^i, b, b^i, c, c^i, m, \mu^i)_{i=1, \dots, d}$ such that F and R are of the Lévy-Khintchine form.*

$$F(u) = \frac{1}{2} \langle u, Au \rangle + \langle b, u \rangle - c + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{\langle \xi, u \rangle} - 1 - \langle h(\xi), u \rangle \right) m(d\xi) \quad (\text{A.1})$$

$$R_i(u) = \frac{1}{2} \langle u, A^i u \rangle + \langle b^i, u \rangle - c^i + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{\langle \xi, u \rangle} - 1 - \langle \chi^i(\xi), u \rangle \right) \mu^i(d\xi), \quad (\text{A.2})$$

where A, A^1, \dots, A^d are positive semi-definite real $d \times d$ -matrices; b, b^1, \dots, b^d are \mathbb{R}^d -valued vectors; c, c^1, \dots, c^d are positive non-negative numbers; m and μ^1, \dots, μ^d are Lévy measures on \mathbb{R}^d , and finally h and χ^1, \dots, χ^d are suitably chosen truncation functions for the respective Lévy measures. Furthermore, the generator \mathcal{A} of Y is given by

$$\begin{aligned} \mathcal{A}\varphi(x) = & \frac{1}{2} \sum_{k,l=1}^d \left(A_{kl} + \sum_{i \in I} A_{kl}^i x_i \right) \frac{\partial^2 \varphi(x)}{\partial x_k \partial x_l} \\ & + \langle b + \sum_{i=1}^d b^i x_i, \nabla \varphi(x) \rangle - \left(c + \sum_{i \in I} c^i x_i \right) \varphi(x) \\ & + \int_{D \setminus \{0\}} (\varphi(c + \xi) - \varphi(x) - \langle h(\xi), \nabla \varphi(x) \rangle) m(d\xi) \\ & + \sum_{i \in I} \int_{D \setminus \{0\}} (\varphi(c + \xi) - \varphi(x) - \langle \chi^i(\xi), \nabla \varphi(x) \rangle) x_i \mu^i(d\xi), \end{aligned} \quad (\text{A.3})$$

and ϕ, ψ satisfy the following system of ODEs

$$\partial_t \phi(t, u) = F(\psi(t, u)), \quad \phi(0, u) = 0 \quad (\text{A.4})$$

$$\partial_t \psi(t, u) = R(\psi(t, u)), \quad \psi(0, u) = u. \quad (\text{A.5})$$

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Selbstständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 24.10.2013

Christoph Mainberger